Q1

Quantum Field Theory - Assignment 2

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a) 1. $\frac{d}{dt}\Theta(t-t') = \delta(t-t')$

Integrate both sides from $-\infty$ to arbitrary $\tau \in \mathbb{R}$

$$\int_{-\infty}^{\tau} \frac{d}{dt}\Theta(t-t')\,dt = \int_{-\infty}^{\tau}\delta(t-t')\,dt.$$

Using the fundamental theorem of calculus

LHS

$$\int_{-\infty}^{\tau}\frac{d}{dt}\Theta(t-t')\,dt = \underbrace{\cancel{\lim_{T\to-\infty}\Theta(T-t')}}_{(=0)\text{ since } T-t'<0,\ T\to-\infty} + \Theta(\tau-t').$$

$$= \begin{cases} 1 & \tau > t' \\ 0 & \tau < t' \end{cases}$$

Using definition of $\delta(x)$

RHS

$$\int_{-\infty}^{\tau}\delta(t-t')\,dt = \begin{cases} 1 & \tau > t' \\ 0 & \tau < t' \end{cases} = \Theta(\tau - t') = \underline{\text{LHS}}$$

Since $\underline{\text{LHS} = \text{RHS}}$ holds for any $\tau \in \mathbb{R}$,

$$\frac{d}{dt}\Theta(t-t') = \delta(t-t') \quad \text{as required.}$$

2. Use similar method.

LHS $\int_{-\infty}^{\tau}\frac{d}{dt}\Theta(t'-t)\,dt = +\lim_{T\to-\infty}\Theta(t'-T) - \Theta(t'-\tau)$

$$= 1 - \Theta(t'-\tau) = \begin{cases} 0 & t' > \tau\ (\tau< \\ -1 & t' < \tau\ (\tau> \end{cases}$$

RHS

$$-\int_{-\infty}^{\tau}\delta(t-t')\,dt = \begin{cases} -1 & \tau > t' \\ 0 & \tau < t' \end{cases} = \underline{\text{LHS}}.$$

since LHS = RHS holds for all $\tau \in \mathbb{R}$, $\frac{d}{dt}\Theta(t'-t) = -\delta(t-t')$ as required.

NOTE: For integrals, could have used $\underline{\text{any}}$ $T < t'$, as the lower

Q1/ continued

b) $$G_F(t-t') = \frac{-i}{2\omega} e^{-i\omega(t-t')}\Theta(t-t') - \frac{i}{2\omega} e^{i\omega(t-t')}\Theta(t'-$$

$$\frac{d}{dt} G_F(t-t') = -\frac{1}{2} e^{-i\omega(t-t')}\Theta(t-t') - \frac{i}{2\omega} e^{-i\omega(t-t')}\frac{d}{dt}\Theta(t-t')$$
$$+ \frac{1}{2} e^{i\omega(t-t')}\Theta(t'-t) - \frac{i}{2\omega} e^{i\omega(t-t')}\frac{d}{dt}\Theta(t'-t)$$

Use the results in a)

$$\Rightarrow \frac{d}{dt} G_F(t-t') = \frac{1}{2}\left[-e^{-i\omega(t-t')}\Theta(t-t') + e^{i\omega(t-t')}\Theta(t'-t)\right.$$
$$\cancel{-\frac{i}{\omega} e^{-i\omega(t-t')}\delta(t-t') + \frac{i}{\omega} e^{i\omega(t-t')}\delta(t-t}$$

using the "hint"; $e^{-i\omega(t-t')}\Big|_{t=t'} = e^{i\omega(t-t')}\Big|_{t=t'} = e^0 = 1$

$$\frac{d}{dt} G_F(t-t') = \frac{1}{2}\left[e^{i\omega(t-t')}\Theta(t'-t) - e^{-i\omega(t-t')}\Theta(t-t')\right]$$

Differentiate again

$$\frac{d^2}{dt^2} G_F(t-t') = \frac{1}{2}\left[i\omega e^{i\omega(t-t')}\Theta(t'-t) + i\omega e^{-i\omega(t-t')}\Theta(t-t')\right.$$
$$\left. - \underbrace{\cancel{e^{i\omega(t-t')}}}_{(1)}\Big|_{t=t'}\delta(t-t') - \underbrace{\cancel{e^{-i\omega(t-t')}}}_{(1)}\Big|_{t=t'}\delta(t-t')\right]$$

$$= +\frac{i\omega}{2} e^{-i\omega(t-t')}\Theta(t-t') + \frac{i\omega}{2} e^{i\omega(t-t')}\Theta(t'-t)$$
$$- \delta(t-t')$$

$$= -\omega^2 G_F(t-t') - \delta(t-t')$$

$$\Rightarrow \boxed{\left(-\frac{d^2}{dt^2} - \omega^2\right) G_F(t-t') = \delta(t-t')}$$ As required.

Q2/

a) $$\frac{d}{dt}\hat{O}_H(t) = \frac{i}{\hbar}\left[\hat{H}, \hat{O}_H(t)\right]$$

Consider $\hat{H} = \frac{1}{2}\hbar\omega(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})$ in Schrödinger picture

$$\left[\hat{H}, \hat{H}\right] = \frac{1}{4}\hbar^2\omega^2\left[\cancel{(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})^2 - (\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})^2}\right] = 0.$$

In the Heisenberg picture:

$$\hat{H}_H = e^{i\hat{H}t/\hbar}\hat{H}e^{-i\hat{H}t/\hbar} = \cancel{e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}}\hat{H}$$

since $\hat{H}$ commutes with itself

$e^{-i\hat{H}t/\hbar}$ contains infinite sum of $\hat{H}$ ter

$$\Rightarrow \hat{H} = e^{i\hat{H}t/\hbar}\frac{1}{2}\hbar\omega(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})e^{-i\hat{H}t/\hbar}$$

$$= \frac{1}{2}\hbar\omega\left[e^{i\hat{H}t/\hbar}\hat{a}\hat{a}^+e^{-i\hat{H}t/\hbar} + e^{i\hat{H}t/\hbar}\hat{a}^+\hat{a}e^{-i\hat{H}t/\hbar}\right]$$

$$= \frac{1}{2}\hbar\omega\left(\hat{a}_H(t)\hat{a}_H^+(t) + \hat{a}_H^+(t)\hat{a}_H(t)\right)$$

since $\hat{a}_H(t)\hat{a}_H^+(t) = e^{i\hat{H}t/\hbar}\hat{a}\,\cancel{e^{-i\hat{H}t/\hbar}e^{i\hat{H}t/\hbar}}\,\hat{a}^+e^{-i\hat{H}t/\hbar}$

$$= e^{i\hat{H}t/\hbar}\hat{a}\hat{a}^+e^{-i\hat{H}t/\hbar}$$

Similar for $\hat{a}_H^+(t)\hat{a}_H(t)$

Now

$$\frac{d}{dt}\hat{a}_H(t) = \frac{i}{\cancel{\hbar}} \times \frac{1}{2}\cancel{\hbar}\omega\left[\hat{a}_H\hat{a}_H^+ + \hat{a}_H^+\hat{a}_H, \hat{a}_H\right].$$

$$= \frac{i\omega}{2}\left(\hat{a}_H\hat{a}_H^+\hat{a}_H + \hat{a}_H^+\hat{a}_H^2 - \hat{a}_H^2\hat{a}_H^+ - \hat{a}_H\hat{a}_H^+\hat{a}\right.$$

$$= \frac{i\omega}{2}\left(\left[\hat{a}_H\hat{a}^+, \hat{a}_H\right] + \left[\hat{a}_H^+\hat{a}, \hat{a}_H\right]\right)$$

$$\Rightarrow \quad = \frac{i\omega}{2}\Big(\hat{a}_H [\hat{a}_H^+, \hat{a}_H] + \cancel{[\hat{a}_H, \hat{a}_H] \hat{a}_H^+} + \cancel{\hat{a}_H^+ [\hat{a}_H, \hat{a}_H]}$$

$$+ [\hat{a}_H^+, \hat{a}_H] \hat{a}_H \Big).$$

$$\frac{d}{dt}\hat{a}_H = -i\omega \hat{a}_H .$$

$$\frac{d}{dt}\hat{a}_H = \frac{d}{dt}\Big(e^{i\hat{H}t/\hbar} \hat{a} e^{-i\hat{H}t/\hbar}\Big) = \frac{i\hat{H}}{\hbar} e^{i\hat{H}t/\hbar} \hat{a} e^{-i\hat{H}t/\hbar}$$

$$- \frac{i}{\hbar} e^{i\hat{H}t/\hbar} \hat{a} e^{-i\hat{H}t/\hbar} \hat{H}$$

$$\frac{d}{dt}\hat{a}_H = -i\omega \hat{a} e^{-i\omega t}$$

$$\Rightarrow \boxed{\hat{a}_H = \hat{a} e^{-i\omega t}.}$$

Q2/

b) $\hat{O}_H(t) = e^{i\hat{H}t/\hbar}\,\hat{O}\,e^{-i\hat{H}t/\hbar}$

consider

$$\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}}\; e^{i\hat{H}t/\hbar}(\hat{a}+\hat{a}^+)e^{-i\hat{H}t/\hbar}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[e^{i\hat{H}t/\hbar}\hat{a}e^{-i\hat{H}t/\hbar} + e^{i\hat{H}t/\hbar}\hat{a}^+e^{-i\hat{H}t/\hbar}\right]$$

$$\therefore\; \hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_H(t) + \hat{a}^+_H(t)\right)$$

Now, $\hat{a}_H(t) = \hat{a}e^{-i\omega t}$; $(AB)^+ = B^+A^+$

$$\Rightarrow \hat{a}^+_H(t) = \hat{a}^+e^{+i\omega t}$$

$$\hat{x}_H(t_1)|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\cancel{e^{-i\omega t_1}\hat{a}|0\rangle} + e^{i\omega t_1}\hat{a}^+|0\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,e^{i\omega t_1}|1\rangle$$

Then

$$\hat{x}_H(t_2)\hat{x}_H(t_1)|0\rangle = \frac{\hbar}{2m\omega}e^{i\omega t_1}\left[\hat{a}_H(t_2)|1\rangle + \hat{a}^+_H(t_2)|1\rangle\right]$$

$$= \frac{\hbar}{2m\omega}e^{i\omega t_1}\left[e^{-i\omega t_2}|0\rangle + e^{i\omega t_2}|2\rangle\right]$$

Taking the inner product

$$\langle 0|\hat{x}_H(t_2)\hat{x}_H(t_1)|0\rangle = \frac{\hbar}{2m\omega}e^{i\omega t_1}\left(e^{-i\omega t_2}\langle 0|0\rangle + \cancel{e^{i\omega t_2}\langle 0|2\rangle}\right)$$

$$\boxed{\langle 0|\hat{x}_H(t_2)\hat{x}_H(t_1)|0\rangle = \frac{\hbar}{2m\omega}e^{i\omega(t_1-t_2)}}$$

Since $\langle n|m\rangle = \delta_{nm}$

$|n\rangle$ are energy eigenstates.

Q2/

c) $$\langle 0|\hat{x}_H(t_2)\,\hat{x}_H(t_1)|0\rangle = \frac{\hbar}{2m\omega}e^{-i\omega(t_2-t_1)}$$

for $t_2 > t_1$.

Now,

$$T\left(\hat{x}_H(t_2)\,\hat{x}_H(t_1)\right) = \begin{cases} \hat{x}_H(t_2)\,\hat{x}_H(t_1) & t_2 > t_1 \\ \hat{x}_H(t_1)\,\hat{x}_H(t_2) & t_1 > t_2 \end{cases}$$

$\Rightarrow$

$$\langle 0|T\left(\hat{x}_H(t_2)\hat{x}_H(t_1)\right)|0\rangle = \frac{\hbar}{2m\omega}\begin{cases} e^{-i\omega(t_2-t_1)} & t_2 > t \\ e^{+i\omega(t_2-t_1)} & t_2 > \end{cases}$$

$\left(\equiv e^{-i\omega(t_1-t_2)}\right)$

This can be written in terms of Heavy side functions.

$$= \frac{\hbar}{2m\omega}\left[e^{-i\omega(t_2-t_1)}\,\theta(t_2-t_1) + e^{i\omega(t_2-t_1)}\,\theta(t_1-t\right.$$

By comparing with Q1 b)

$$\langle 0|T\left(\hat{x}_H(t_2)\,\hat{x}_H(t_1)\right)|0\rangle = \frac{-\hbar}{mi}\left[-\frac{i}{2\omega}e^{-i\omega(t_2-t_1)}\theta(t_2-t_1) - \frac{i}{2\omega}e^{i\omega(t_2-t_1)}\theta(t_1-t_2)\right]$$

$$= +\frac{i\hbar}{m}G_F(t_2-t_1)$$

As required.

[I seem to have t_1 & t_2 around the wrong way, but just defining $t_2 \Leftrightarrow t_1$ gets the same result $G_F(t_1-t_2)$]

Q2/

d) $$\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a}_H(t) + \hat{a}_H^+(t) \right).$$

$$\hat{a}_H(t) = \hat{a}\, e^{-i\omega t} \qquad \hat{a}_H^+(t) = \hat{a}^+ e^{i\omega t}$$

$$\frac{d}{dt}\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}} \left(\frac{d}{dt}\hat{a}_H(t) + \frac{d}{dt}\hat{a}_H^+(t) \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(-i\omega\, \hat{a}_H(t) + i\omega\, \hat{a}_H^+(t) \right)$$

$$\frac{d^2}{dt^2}\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}} \left(-\omega^2 \hat{a}_H(t) - \omega^2 \hat{a}_H^+(t) \right)$$

$$= -\omega^2 \hat{x}_H(t).$$

$$\Rightarrow \boxed{\left(-\frac{d^2}{dt^2} - \omega^2 \right) \hat{x}_H(t) = 0.}$$ As required.