

Number Representations and Precision in Vector Graphics

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Abstract

Early document formats such as PostScript were motivated by a desire to print text and visual information onto a static paper medium. Although documents are increasingly viewed digitally, modern standards including PDF and SVG are still largely based upon this model. Digital document viewers are able to scale a subregion of the document to fit the display. However, coordinates of graphics primitives are typically represented with IEEE-754 floating point numbers. This places limits on the precision with which primitives in the document can be specified and rendered.

We have implemented a minimal SVG viewer, with which we have compared a number of approaches to achieving arbitrary precision document formats. We demonstrate the trade off between performance and precision with alternative number representations including arbitrary precision floats, rationals, and IEEE-754 fixed precision floats. We also consider approaches to increasing the precision that can be attained with IEEE-754 floats.

Keywords: *document formats, precision, floating point, vector images, graphics, OpenGL, SDL2, PostScript, PDF, T_EX, SVG, HTML5, Javascript*

Note: This report is best viewed digitally as a PDF. The digital version is available at <http://szmoore.net/ipdf/sam/thesis.pdf>

Contents

1	Introduction	1
2	Literature Review	2
2.1	Raster and Vector Graphics	2
2.2	Rendering Vector Primitives	3
2.2.1	Straight Lines	3
2.2.2	Bézier Splines	4
2.2.3	Fonts	6
2.3	Coordinate Systems and Transformations	6
2.3.1	View Transformations	7
2.4	Precision Specified by Document Standards	8
2.4.1	PostScript	8
2.4.2	PDF	8
2.4.3	T _E X and METAFONT	8
2.4.4	SVG	8
2.4.5	Javascript	8
2.5	Fixed Point and Integer Number Representations	9
2.6	Floating Point Number Representations	9
2.6.1	Visualisation of Floating Point Representation	10
2.6.2	Floating Point Operations	11

2.6.3	Arbitrary Precision Floating Point Numbers	11
2.7	Rational Number Representations	12
2.8	Floating Point Operations on the CPU and GPU	12
3	Methods and Design	13
3.1	Structure of Software	13
3.1.1	Document Structure	13
3.1.2	Rendering Process	13
3.1.3	Interactivity	13
3.1.4	Version Control	13
3.2	Approaches to Arbitrary Precision	14
3.2.1	Naïve Approach	14
3.2.2	Intermediate Coordinate Systems	14
3.2.3	Quadtree Document Division	14
3.3	Libraries Used	14
4	Results and Discussion	15
4.1	Qualitative Rendering Accuracy	15
4.1.1	Applying the view transformation directly	15
4.1.2	Applying cumulative transformations to all Béziers	15
4.1.3	Applying cumulative transformations to Paths	16
4.2	Quantitative Measurements of Rendering Accuracy	16
4.2.1	Names of programs in figures	17
4.2.2	Precision for Fixed View	17
4.2.3	Accumulated error after changing the View	18
4.3	Performance Measurements whilst Rendering	18
5	Conclusion	20
	References	23

List of Figures

2.1	Original Vector and Raster Images	2
2.2	Scaled Vector and Raster Images	3
2.3	Rasterising a Straight Line	4
2.4	Constructing a Spline from two cubic Béziers (a) Showing the Control Points (b) Representations in SVG and PostScript (c) Rendered Spline	5

2.5	a) Vector glyph for the letter Z b) Screenshot showing Bézier control points in Inkscape . . .	6
2.6	Illustration of view transformation (2.7)	7
2.7	Positive 8-Bit Number Representations	10
2.8	Difference between successive numbers	11
4.1	The vector image from Figure 2.1 under two different scales	15
4.2	The effect of applying cumulative transformations to all Béziers	16
4.3	Effect of cumulative transformations applied to Paths a) Path bounds represented using floats b) Path bounds represented using Rationals	16
4.4	Effect of applying (2.7) to a grid of lines separated by 1 pixel a) Near origin (denormals) b), c), d) Increasing the exponent of (v_x, v_y) by 1	17
4.5	Loss of precision of the grid	18
4.6	Error in the coordinates of the grid	18
4.7	Memory used per Rational whilst zooming in	19
4.8	Time taken to scale about the fixed point	19

1. Introduction

Early electronic document formats such as PostScript were motivated by a need to print documents onto a paper medium. In the PostScript standard, this led to a model of the document as a program; a series of instructions to be executed by an interpreter which would result in “ink” being placed on “pages” of a fixed size[3]. The ubiquitous Portable Document Format (PDF) standard provides many enhancements to PostScript taking into account desktop publishing requirements[4], but it is still fundamentally based on the same imaging model[5]. This idea of a document as a static “page” has led to limitations on what could be achieved with a digital document viewers [6].

The emergence of the internet, web browsers, XML/HTML, JavaScript and related technologies has seen a revolution in the ways in which information can be presented digitally, and the PDF standard itself has begun to move beyond static text and figures[6, 7]. However, the popular document formats are still designed with the intention of showing information at either a single, fixed level of detail, or a small range of levels.

As most digital display devices are smaller than physical paper medium, all useful viewers are able to “zoom” to a subset of the document. Vector graphics formats including PostScript, PDF and SVG support rasterisation at different zoom levels[3, 5, 8], but the use of fixed precision floating point numbers causes problems due to imprecision either far from the origin, or at a high level of detail[9, 10].

There are many possible applications for documents in which precision is unlimited. Several areas of use include: visualisation of extremely large or infinite data sets; visualisation of high precision numerical computations; digital artwork; computer aided design; and maps.

In collaboration with Gow[] we have implemented a proof of concept document viewer compatible with a subset of the SVG standard, which has allowed us to explore the limitations of floating point arithmetic and possible approaches to achieving arbitrary precision document formats. Using the Rational representation of the GNU Multiple Precision (GMP) library[11] we are able to implement correct rendering of SVG test images separated by arbitrary distances. We present measurements of rendering accuracy and performance for our implementation.

2. Literature Review

2.1 Raster and Vector Graphics

At a fundamental level everything that is seen on a display device is represented as either a vector or raster image. These images can be stored as stand alone documents or embedded within a more complex document format capable of containing many other types of information.

A raster image's structure closely matches it's representation as shown on modern display hardware; the image is represented as a grid of filled square "pixels". Each pixel is considered to be a filled square of the same size and contains information describing its colour. This representation is simple and also well suited to storing images as produced by cameras and scanners. The drawback of raster images is that by their very nature there can only be one level of detail; this is illustrated in Figures 2.1 and 2.2.

A vector image contains information about the positioning and shading of geometric shapes. To display this image on modern display hardware, coordinates are transformed according to the view and then the image is converted into a raster like representation. Whilst the raster image merely appears to contain edges, the vector image actually contains information about these edges, meaning they can be displayed "infinitely sharply" at any level of detail — or they could be if the coordinates are stored with enough precision (see Section ??).

Figures 2.1 and 2.2 illustrate the advantage of vector formats by comparing raster and vector images in a similar way to Worth and Packard[12]. On the right is a raster image which should be recognisable as an animal defined by fairly sharp edges. Figure 2.2 shows how these edges appear jagged when scaled. There is no information in the original image as to what should be displayed at a larger size, so each square shaped pixel is simply increased in size. A blurring effect will probably be visible in most PDF viewers; the software has attempted to make the "edge" appear more realistic using a technique called "antialiasing"¹.

The left side of the Figures are a vector image. When scaled, the edges maintain a smooth appearance which is limited by the resolution of the display rather than the image itself.

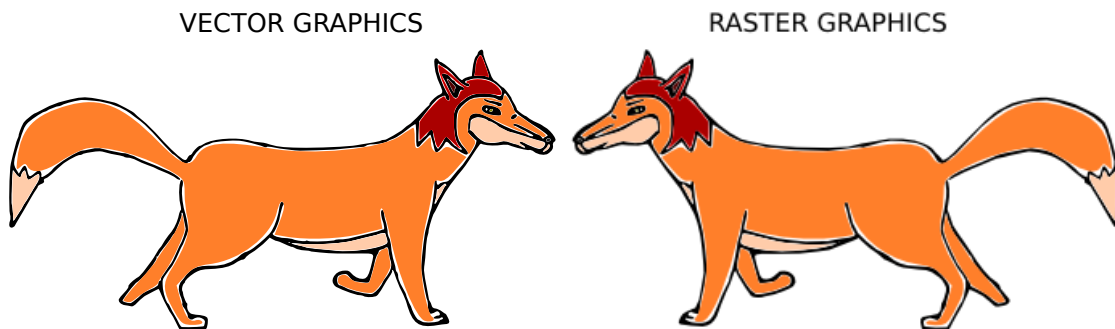


Figure 2.1: Original Vector and Raster Images

¹We recommend disabling this if your PDF viewer supports it

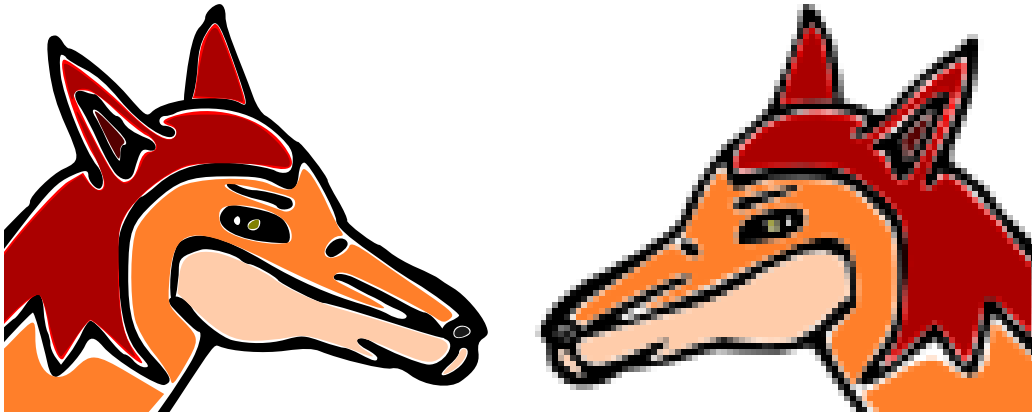


Figure 2.2: Scaled Vector and Raster Images

2.2 Rendering Vector Primitives

Hearn and Baker’s textbook “Computer Graphics”[13] gives a comprehensive overview of graphics from physical display technologies through fundamental drawing algorithms to popular graphics APIs. This section will examine algorithms for drawing two dimensional geometric primitives on raster displays as discussed in “Computer Graphics” and the relevant literature. This section is by no means a comprehensive survey of the literature but intends to provide some idea of the computations which are required to render a document.

It is of some historical significance that vector display devices were popular during the 70s and 80s, and papers oriented towards drawing on these devices can be found[14]. Whilst curves can be drawn at high resolution on vector displays, a major disadvantage was shading[15]; by the early 90s the vast majority of computer displays were raster based[13].

2.2.1 Straight Lines

It is well known that in cartesian coordinates, a line between points (x_1, y_1) and (x_2, y_2) , can be described by:

$$y(x) = mx + c \quad \text{on } x \in [x_1, x_2] \text{ for } m = \frac{(y_2 - y_1)}{(x_2 - x_1)} \text{ and } c = y_1 - mx_1 \quad (2.1)$$

On a raster display, only points (x, y) with integer coordinates can be displayed; however m will generally not be an integer. Thus a straight forward use of Equation 2.1 will require costly floating point operations and rounding (See Section??). Modifications based on computing steps Δx and Δy eliminate the multiplication but are still less than ideal in terms of performance[13].

It should be noted that algorithms for drawing lines can be based upon sampling $y(x)$ only if $|m| \leq 1$; otherwise sampling at every integer x coordinate would leave gaps in the line because $\Delta y > 1$. Line drawing algorithms can be trivially adopted to sample $x(y)$ if $|m| > 1$.

Bresenham’s Line Algorithm was developed in 1965 with the motivation of controlling a partic-

ular mechanical plotter in use at the time[16]. The plotter’s motion was confined to move between discrete positions on a grid one cell at a time, horizontally, vertically or diagonally. As a result, the algorithm presented by Bresenham requires only integer addition and subtraction, and it is easily adopted for drawing pixels on a raster display. Because integer operations are exact, only an error in the calculation of the line end points will affect the rendering.

In Figure 2.3 a) and b) we illustrate the rasterisation of a line width a single pixel width. The path followed by Bresenham’s algorithm is shown. It can be seen that the pixels which are more than half filled by the line are set by the algorithm. This causes a jagged effect called aliasing which is particularly noticeable on low resolution displays. From a signal processing point of view this can be understood as due to the sampling of a continuous signal on a discrete grid[17].

Figure 2.3 c) shows an (idealised) antialiased rendering of the line. The pixel intensity has been set to the average of the line and background colours over that pixel. Such an ideal implementation would be impractically computationally expensive on real devices[18]. In 1991 Wu introduced an algorithm for drawing approximately antialiased lines which, while equivalent in results to existing algorithms by Fujimoto and Iwata, set the state of the art in performance[17]². .

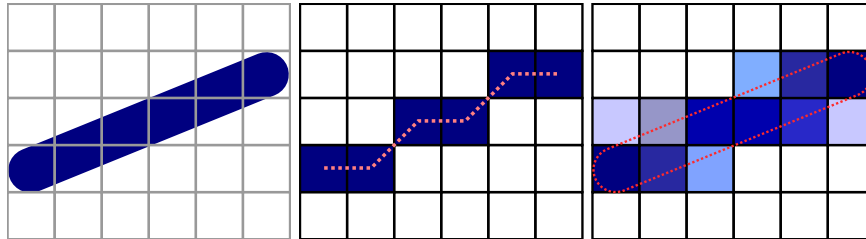


Figure 2.3: Rasterising a Straight Line

a) Before Rasterisation b) Bresenham’s Algorithm c) Anti-aliased Line (Idealised)

2.2.2 Bézier Splines

Splines are continuous curves formed from piecewise polynomial segments. A polynomial of n th degree is defined by n constants $\{a_0, a_1, \dots, a_n\}$ and:

$$y(x) = \sum_{k=0}^n a_k x^k \quad (2.2)$$

Cubic and Quadratic Bézier Splines are used to define curved paths in the PostScript[3], PDF[5] and SVG[8] standards which we will discuss in Section ???. Cubic Béziers are also used to define vector fonts for rendering text in these standards and the \TeX typesetting language [19, 20]. Although he did not derive the mathematics, the usefulness of Bézier curves was realised by Pierre Bézier who used them in the 1960s for the computer aided design of automobile bodies[21].

A Bézier Curve of degree n is defined by n “control points” $\{P_0, \dots, P_n\}$. Points $P(t) = (x(t), y(t))$

²Techniques for antialiasing primitives other than straight lines are discussed in some detail in Chapter 4 of “Computer Graphics” [13]

along the curve are defined by:

$$P(t) = \sum_{j=0}^n B_j^n(t) P_j \tag{2.3}$$

Where $t \in [0, 1]$ is a control parameter. The polynomials $B_j^n(t)$ are Bernstein Basis Polynomials which are defined as:

$$B_j^n(t) = \binom{n}{j} t^j (1-t)^{n-j} \quad j = 0, 1, \dots, n \tag{2.4}$$

Where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$ (The Binomial Coefficients) (2.5)

From these definitions it should be apparent that in all cases, $P(0) = P_0$ and $P(1) = P_n$. An $n = 1$ Bézier Curve is a straight line.

Algorithms for rendering Bézier's may simply sample $P(t)$ for sufficiently many values of t — enough so that the spacing between successive points is always less than one pixel distance. Alternately, a smaller number of points may be sampled with the resulting points connected by straight lines using one of the algorithms discussed in Section ??.

De Casteljaeu's algorithm of 1959 is often used for decomposing Béziars into line segments[13, 19]. This algorithm subdivides the original curve with n control points $\{P_0, \dots, P_n\}$ into 2 halves, each with n control points: $\{Q_0, \dots, Q_n\}$ and $\{R_0, \dots, R_n\}$; when iterated, the produced points will converge to $P(t)$. As a tensor equation this subdivision can be expressed as[22]:

$$Q_i = \binom{\binom{n}{j}}{2^j} P_i \text{ and } R_i = \binom{\binom{n-j}{n-k}}{2^{n-j}} P_i \tag{2.6}$$

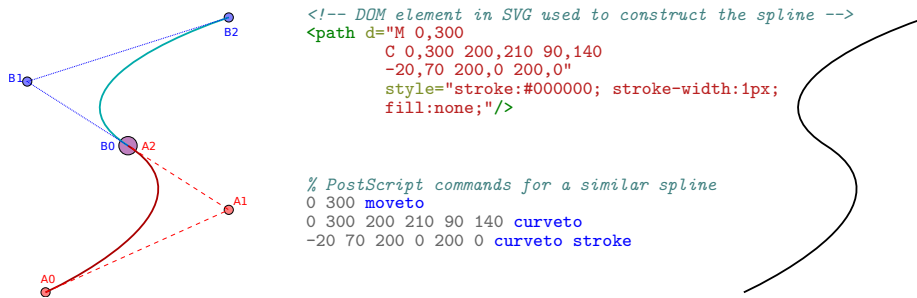


Figure 2.4: Constructing a Spline from two cubic Béziars
 (a) Showing the Control Points (b) Representations in SVG and PostScript (c) Rendered Spline

2.2.3 Fonts

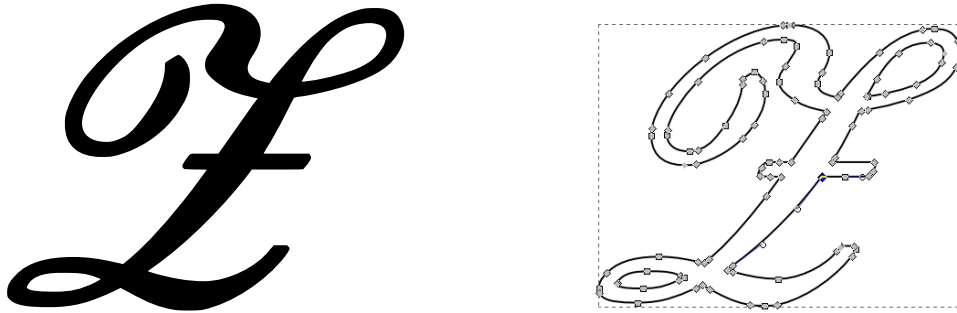


Figure 2.5: a) Vector glyph for the letter Z b) Screenshot showing Bézier control points in Inkscape

The term “font” refers to a set of images used to represent text on a graphical display. In 1983, Donald Knuth published “The METAFONT Book” which described a vector approach to specifying fonts and a program for creating these fonts[19]. Previously, only rasterised font images (glyphs) were popular; as can be seen from the zooming in Figure 2.2 this can be problematic given the prevalence of textual information at different scales and on different resolution displays.

Knuth used Bézier Cubic Splines to define “pleasing” curves in METAFONT, and this approach is still used in modern vector fonts. Since the paths used to render an individual glyph are used far more commonly than general curves, document formats do not require such curves to be specified in situ, but allow for a choice between a number of internal fonts or externally specified fonts. In the case of Knuth’s typesetting language \TeX , fonts were intended to be created using METAFONT[19]. Figure 2.5 shows a \mathcal{Z} (script Z) produced by \LaTeX with Bézier cubics identified.

2.3 Coordinate Systems and Transformations

Basic vector primitives composed of Béziers may be rendered using only integer operations, once the starting and ending positions are rounded to the nearest pixel.

However, a complete document will contain many such primitives which in general cannot all be shown on a display at once. A “View” rectangle can be defined to represent the size of the display relative to the document. To interact with the document a user can change this view through scaling or translating with the mouse.

Primitives which are contained within the view rectangle will be visible on the display. This involves the transformation from coordinates within the document to relative coordinates within the view rectangle as illustrated in Figure 2.6. A point (X, Y) in the document will transform to a point (S_X, S_Y) in the display by:

$$S_X = \frac{X - V_x}{V_w} \quad S_Y = \frac{Y - V_y}{V_h} \quad (2.7)$$

Where (V_x, V_y) are the coordinates of the top left corner and (V_w, V_h) are the dimensions of the view rectangle.

The transformation may also be written as a 3x3 matrix \mathbf{V} if we introduce a third coordinate $z = 1$

$$\begin{pmatrix} S_X \\ S_Y \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{V_w} & 0 & \frac{V_x}{V_w} \\ 0 & \frac{1}{V_h} & \frac{V_y}{V_h} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} \quad (2.8)$$

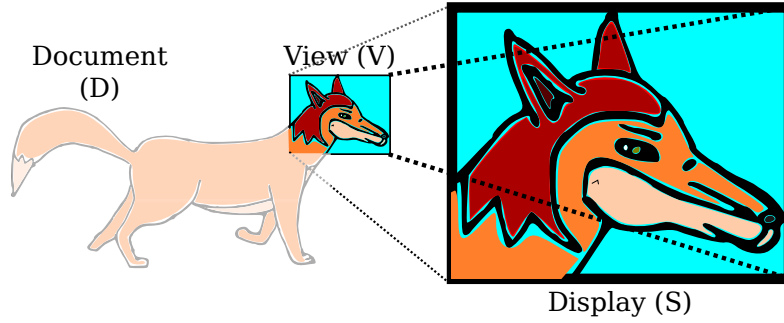


Figure 2.6: Illustration of view transformation (2.7)

2.3.1 View Transformations

Moving the mouse³ by a distance $(\Delta x, \Delta y)$ relative to the size of the view should translate it by the same amount[]:

$$V_x \rightarrow V_x + \Delta x \quad (2.9)$$

$$V_y \rightarrow V_y + \Delta y \quad (2.10)$$

The document can be scaled by a factor of s about a point (x_0, y_0) specified relative to the view (such as the position of the mouse cursor)[]:

$$V_x \rightarrow V_x + x_0 V_w (1 - s) \quad (2.11)$$

$$V_y \rightarrow V_y + y_0 V_h (1 - s) \quad (2.12)$$

$$V_w \rightarrow s V_w \quad (2.13)$$

$$V_h \rightarrow s V_h \quad (2.14)$$

The effect of this transformation is that, measured relative to the view rectangle, the distance of primitives with coordinates (x, y) to the point (x_0, y_0) will decrease by a factor of s . For $s < 1$ the operation is “zooming out” and for $s > 1$, “zooming in”.

³or on a touch screen, swiping the screen

2.4 Precision Specified by Document Standards

2.4.1 PostScript

The PostScript reference describes a “Real” object for representing coordinates and values as follows: “Real objects approximate mathematical real numbers within a much larger interval, but with limited precision; they are implemented as floating-point numbers”[3]. There is no reference to the precision of mathematical operations, but the implementation limits *suggest* a range of $\pm 10^{38}$ “approximate” and the smallest values not rounded to zero are $\pm 10^{-38}$ “approximate”.

2.4.2 PDF

PDF defines “Real” objects in a similar way to PostScript, but suggests a range of $\pm 3.403 \times 10^{38}$ and smallest non-zero values of $\pm 1.175 \times 10^{38}$ [5]. A note in the PDF 1.7 manual mentions that Acrobat 6 now uses IEEE-754 single precision floats, but “previous versions used 32-bit fixed point numbers” and “... Acrobat 6 still converts floating-point numbers to fixed point for some components”.

2.4.3 T_EX and METAFONT

In “The METAFONT book” Knuth appears to describe coordinates as fixed point numbers: “The computer works internally with coordinates that are integer multiples of $\frac{1}{65536} \approx 0.00002$ of the width of a pixel”[19].⁴ There is no mention of precision in “The T_EX book”. In 2007 Beebe claimed that T_EX uses a 14.16 fixed point encoding, and that this was due to the lack of standardised floating point arithmetic on computers at the time; a problem that the IEEE-754 was designed to solve[23]. Beebe also suggested that T_EX and METAFONT could now be modified to use IEEE-754 arithmetic.

2.4.4 SVG

The SVG standard specifies a minimum precision equivalent to that of “single precision floats” (presumably referring to IEEE-754) with a range of $-3.4\text{e}+38\text{F}$ to $+3.4\text{e}+38\text{F}$, and states “It is recommended that higher precision floating point storage and computation be performed on operations such as coordinate system transformations to provide the best possible precision and to prevent round-off errors.”[8] An SVG Viewer may refer to itself as “High Quality” if it uses a minimum of “double precision” floats.

2.4.5 Javascript

We include Javascript here due to its relation with the SVG, HTML5 and PDF standards. According to the ECMA-262 standard, “The Number type has exactly 18437736874454810627 (that is, $2^{64} - 5 \cdot 3 + 3$) values, representing the double-precision 64-bit format IEEE 754 values as specified in the IEEE Standard for Binary Floating-Point Arithmetic”[24]. The Number type does differ

⁴This corresponds to using 16 bits for the fractional component of a fixed point representation

slightly from IEEE-754 in that there is only a single valid representation of “Not a Number” (NaN). The EMCA-262 does not define an “integer” representation.

2.5 Fixed Point and Integer Number Representations

A positive real number z may be written as the sum of smaller integers “digits” d_i multiplied by powers of a base β .

$$z = d_0\beta^0 + d_1\beta^1 + d_2\beta^2 + \dots = \sum_{i=-\infty}^{\infty} d_i\beta^i \quad (2.15)$$

Where each digit $d_i < \beta$. A set of β unique symbols are used to represent values of d_i . A separate sign ‘-’ can be used to represent negative reals using equation (2.15).

To express a real number using equation (2.15) in practice we are limited to a finite number of terms between $i = -m$ and $i = n$. Fixed point representations are capable of representing a discrete set of numbers $0 \leq |z| \leq \beta^{n+1} - \beta^{-m}$ separated by $\Delta z = \beta^{-m} \leq 1$. In the case $m = 0$, only integers can be represented.

Example integer representation in base 10 (decimal) and base 2 (binary):

$$\begin{aligned} 5682_{10} &= 5 \times 10^3 + 6 \times 10^2 + 8 \times 10^1 + 2 \times 10^0 \\ 1011000110010_2 &= 1 \times 2^{12} + 0 \times 2^{11} + \dots + 0 \times 2^0 \end{aligned}$$

2.6 Floating Point Number Representations

Whilst a Fixed Point representation keeps the “point” (the location considered to be $i = 0$ in (2.15)) at the same position in a string of bits, Floating point representations can be thought of as scientific notation; an “exponent” and fixed point value are encoded, with multiplication by the exponent moving the position of the point.

The use of floating point arithmetic in computer systems was pioneered by Knuth, Goldberg, Dekker, and others[25], but modern systems are largely compatible with the IEEE-754 standard pioneered by William Kahan in 1985 [26] and revised (also with contributions from Kahan) in 2008[27].

A floating point number x is commonly represented by a tuple of values (s, e, m) in base B as[25, 28]: $x = (-1)^s \times m \times B^e$

Where s is the sign and may be zero or one, m is commonly called the “mantissa” and e is the exponent. Whilst e is an integer in some range $\pm e_{max}$, the mantissa m is a fixed point value in the range $0 < m < B$. The choice of base $B = 2$ in the original IEEE-754 standard matches the nature of modern hardware. It has also been found that this base in general gives the smallest rounding errors[25].

The IEEE-754 encoding of s , e and m requires a fixed number of continuous bits dedicated to each value. Originally two encodings were defined: binary32 and binary64. s is always encoded in

a single leading bit, whilst (8,23) and (11,53) bits are used for the (exponent, mantissa) encodings respectively.

The encoding of m in the IEEE-754 standard is not exactly equivalent to a fixed point value. By assuming an implicit leading bit (ie: restricting $1 \leq m < 2$) except for when $e = 0$, floating point values are guaranteed to have a unique representations; these representations are said to be “normalised”. When $e = 0$ the leading bit is not implied; these representations are called “denormals” because multiple representations may map to the same real value. The idea of using an implicit bit appears to have been considered by Goldberg as early as 1967[29], and it leads to an increase of precision near the origin.

2.6.1 Visualisation of Floating Point Representation

Figure ?? shows the positive real numbers which can be represented exactly by an 8 bit floating point number encoded in the IEEE-754 format. We show two encodings using (1,2,5) and (1,3,4) bits to encode (sign, exponent, mantissa) respectively. For each distinct value of the exponent, the successive floating point representations lie on a straight line with constant slope. As the exponent increases, larger values are represented, but the distance between successive values increases; this can be seen in Figure??. The marked single point discontinuity at 0x10 and 0x20 occur when e leaves the denormalised region and the encoding of m changes. We have also plotted a fixed point representation for comparison; fixed point and integer representations appear on straight lines.

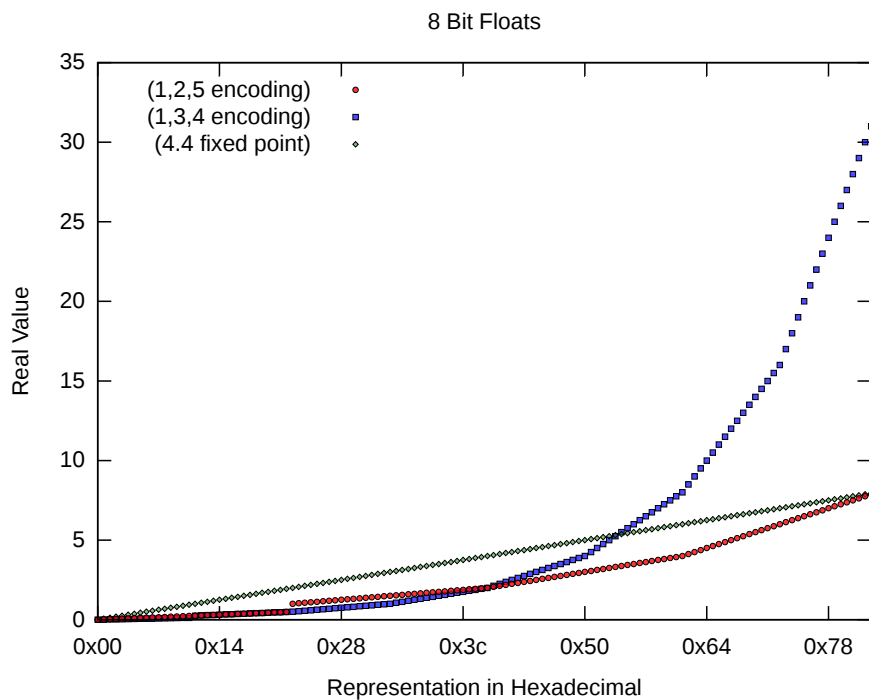


Figure 2.7: Positive 8-Bit Number Representations

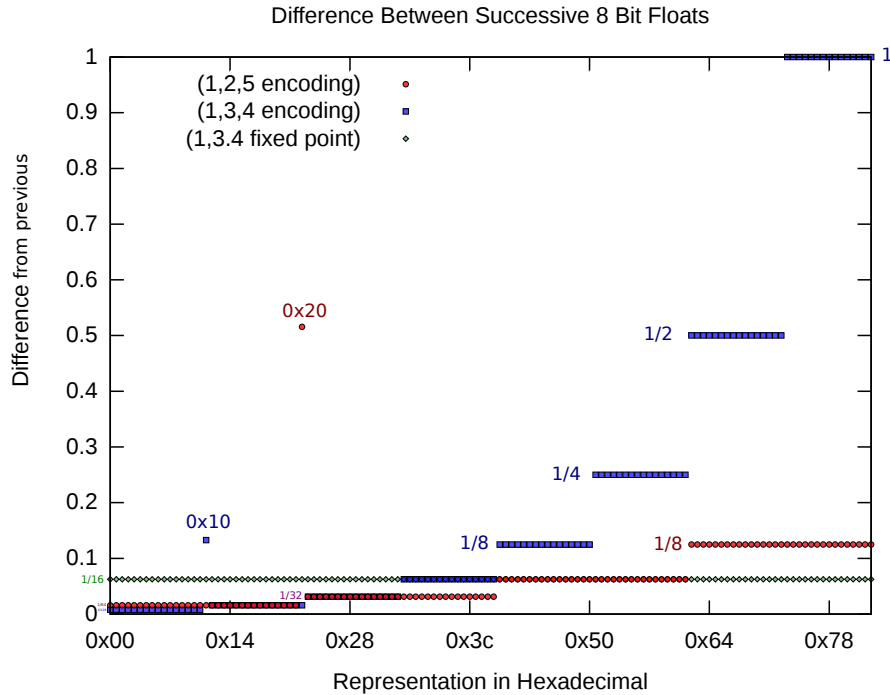


Figure 2.8: Difference between successive numbers

2.6.2 Floating Point Operations

Real values which cannot be represented exactly in a floating point representation must be rounded to the nearest floating point value. The results of a floating point operation will in general be such values and thus there is a rounding error possible in any floating point operation[25].

Referring to Figure ?? it can be seen that the largest possible rounding error is half the distance between successive floats; this means that rounding errors increase as the value to be represented increases. For the result of a particular operation, the maximum possible rounding error can be determined and is commonly expressed in “units in the last place” (ulp), with 1 ulp equivalent to half the distance between successive floats[9].

2.6.3 Arbitrary Precision Floating Point Numbers

Arbitrary precision floating point numbers are implemented in a variety of software libraries which will dynamically allocate extra bits for the exponent or mantissa as required. An example is the GNU MPFR library discussed by Fousse in 2007[30]. Although many arbitrary precision libraries already existed, MPFR intends to be fully compliant with some of the more obscure IEEE-754 requirements such as rounding rules and exceptions.

It is trivial to find real numbers that would require an infinite number of bits to represent exactly (for example, $\frac{1}{3} = 0.333333\dots$). The GMP and MPFR libraries require a fixed (but arbitrarily large) precision be set; although it is possible to increase or decrease the precision of individual numbers as desired.

2.7 Rational Number Representations

$$Q = \frac{N}{D} \tag{2.16}$$

- N and D are arbitrary precision integers

$$N = \sum_{i=0}^S d_i \beta^i \tag{2.17}$$

- d_i are fixed size integers, $\beta = 2^{64}$
- Size S grows as needed
- Operations are always exact
- Implemented by GNU Multiple Precision Library

2.8 Floating Point Operations on the CPU and GPU

FIXME: I feel this section is important but I'm not quite sure where to place it; it could almost work as a paper by itself (in fact I sort of wrote one for it already...)

Traditionally, vector images have been rasterized by the CPU before being sent to a specialised Graphics Processing Unit (GPU) for drawing[13]. Rasterisation of simple primitives such as lines and triangles have been supported directly by GPUs for some time through the OpenGL standard[31]. However complex shapes (including those based on Bézier curves such as font glyphs) must either be rasterised entirely by the CPU or decomposed into simpler primitives that the GPU itself can directly rasterise. There is a significant body of research devoted to improving the performance of rendering such primitives using the latter approach, mostly based around the OpenGL[31] API[32, 33, 34, 35, 36, 37]. Recently Mark Kilgard of the NVIDIA Corporation described an extension to OpenGL for NVIDIA GPUs capable of drawing and shading vector paths[38, 39]. From this development it seems that rasterization of vector graphics may eventually become possible upon the GPU.

It is not entirely clear how well supported the IEEE-754 standard for floating point computation is amongst GPUs⁵. Although the OpenGL API does use IEEE-754 number representations, research by Hillesland and Lastra in 2004 suggested that many GPUs were not internally compliant with the standard[40]. In Section ?? we illustrate how the use of

⁵Informal technical articles are abundant on the internet — Eg: Regarding the Dolphin Wii GPU Emulator: (<https://dolphin-emu.org/blog>) (accessed 2014-05-22)

3. Methods and Design

3.1 Structure of Software

A custom vector graphics viewer called IPDF¹ was implemented for this project in collaboration with David Gow[1]. We will give a brief overview of the features and limitations of this software.

3.1.1 Document Structure

IPDF is built around Objects which can be rendered either on the CPU (directly altering a bitmap and then uploading to the GPU) or the GPU (via the use of OpenGL GLSL shader programs).

Initially only very simple shapes (Rectangles and Circles) were supported, but in order to produce a meaningful demonstration of arbitrary precision viewing, Paths formed from Quadratic or Cubic Béziers as specified by the SVG standard were added. Shading of paths is partially implemented.

3.1.2 Rendering Process

As discussed in Section ?? it is not clear to what extent GPUs comply with the IEEE-754 standard, and arbitrary precision arithmetic is most easily implemented on the CPU; for this reason a CPU renderer and GPU renderer were implemented.

All Objects are represented by their bounding rectangles; in the case of Bézier curves the control points are stored relative to the bounding rectangle. The rendering process involves converting the bounding rectangle coordinates from Document to Screen space as discussed in Section ?. For both GPU and CPU rendering a Bézier in screen space is decomposed into straight lines; the CPU renders these lines using Bresenham’s algorithm.

3.1.3 Interactivity

There are two ways to control the IPDF software; manually through use of keyboard and mouse and a Qt4[2] based control panel, or automatically through a script containing a sequence of commands to transform the view or insert SVGs. In both cases SVGs can be inserted scaled to the current View rectangle.

3.1.4 Version Control

The Git version control system was used to collaborate on this project; the main repository may be viewed at <http://git.ucc.asn.au/?p=ipdf/code.git>

¹The original name “Infinite Precision Document Format” stuck, although the word “Infinite” is highly misleading

3.2 Approaches to Arbitrary Precision

3.2.1 Naïve Approach

A naïve approach would be to replace all floating point operations with arbitrary precision operations, and this was in fact tried in early experiments. This approach requires use of the CPU renderer, as GLSL is restricted to floating point representations. A type definition `Real` on the CPU can be selected at compile time.

Unfortunately truly arbitrary precision number representations were found to be far too inefficient for practical purposes — for example, rendering a frame with GMP Rationals could take up to 60 seconds at the default view.

3.2.2 Intermediate Coordinate Systems

When an object is visible on the screen it is only necessary to render it accurately to within the nearest pixel. As shown in the Results Section ??, introducing an intermediate coordinate system for a large number of objects and applying transformations to this coordinate system instead of individual objects produces the best results both in terms of reduced rounding errors using floating point arithmetic, and reduced number of required arbitrary precision operations.

3.2.3 Quadtree Document Division

An approach identified by Gow[] is to construct intermediate coordinate systems as the user manipulates the view.

3.3 Libraries Used

The libraries used to compile the IPDF software are:

- SDL2 - Simple Direct media Library
 - Used for window management and to obtain an OpenGL context
 - Provides Bitmap handling functions
- Qt4 (optional) — Open source toolkit for Dialog based applications
- OpenGL (4.4) — The standard API for controlling GPUs
- PugiXML — Open source XML parsing library used to implement parsing of SVGs
- GNU Multiple Precision (GMP)
 - Implements arbitrary precision integers, floats, and rationals
 - Highly optimised using CPU specific assembly instructions
- MPFR — built on GMP but ensures IEEE-754 consistent rounding behaviour

4. Results and Discussion

4.1 Qualitative Rendering Accuracy

Our ultimate goal is to be able to insert detail at an arbitrary point in the document. Therefore, we are interested in how the same test SVG would appear when scaled to the view coordinates, as the view coordinates are varied.

4.1.1 Applying the view transformation directly

Figure 4.1 shows the rendering of a vector image¹. Transformation (2.7) is applied to object coordinates with default IEEE-754 rounding behaviour (to nearest). The loss of precision in the second figure is obvious. This is because division by 10^{-6} increases the rounding error in $x - v_x$, by 10^6 , so the total error is of the order 10^6 ulp which is of the order 0.25

TODO: Calculate that properly, shouldn't be hard

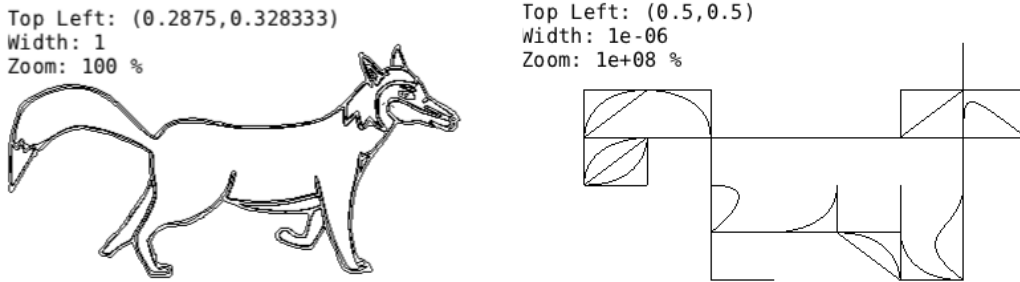


Figure 4.1: The vector image from Figure 2.1 under two different scales

4.1.2 Applying cumulative transformations to all Béziers

Rather than applying (2.7) to object coordinates specified relative to the document, we can store the bounds of objects relative to the view and modify these bounds according to transformations (??) and (??) as the view is changed. This is convenient for an interactive document, as detail is typically added by inserting objects into the document within the view rectangle. As a result this approach makes the rendering of detail added to the document independent of the view coordinates — until the view is moved.

Repeated transformations on the view will cause an accumulated error on the coordinates of object bounds. This is most noticeable when zooming *out* and then back into the document; the object coordinates will gradually underflow and eventually round to zero. An example of this effect is shown in Figure 4.2 b)

¹Unfortunately, since a rendered vector image is a raster image and this figure must be scaled to fit the PDF, the figure as seen here is not a pixel perfect representation of the actual rendering. Most notably, antialiasing effects will be apparent

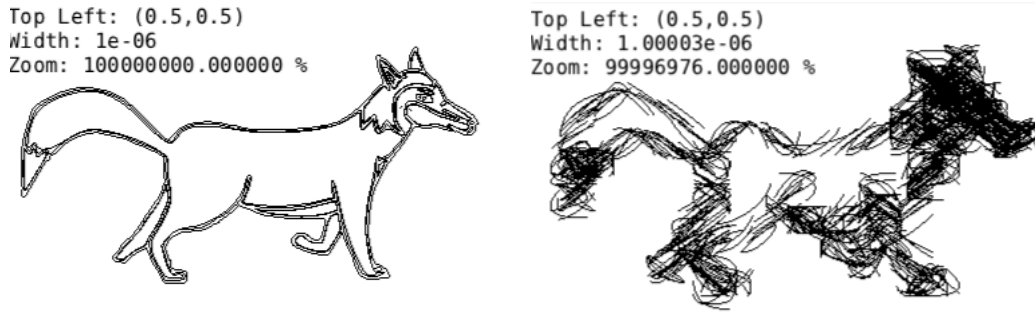


Figure 4.2: The effect of applying cumulative transformations to all Béziers

4.1.3 Applying cumulative transformations to Paths

In Figure 4.1, transformations are applied to the bounds of each Bézier. Figure 4.3 a) shows the effect of introducing an intermediate coordinate system expressing Bézier coordinates relative to the path which contains them. In this case, the rendering of a single path is accurate, but the overall positions of the paths drift as the view is moved.

We can correct this drift whilst maintaining performance by using an arbitrary or high precision number representation to express the coordinates of the paths - but maintaining the floating point coordinates for Bézier curves relative to their path. As we will discuss in Section ??, this offers an acceptable trade off between rendering accuracy and performance.

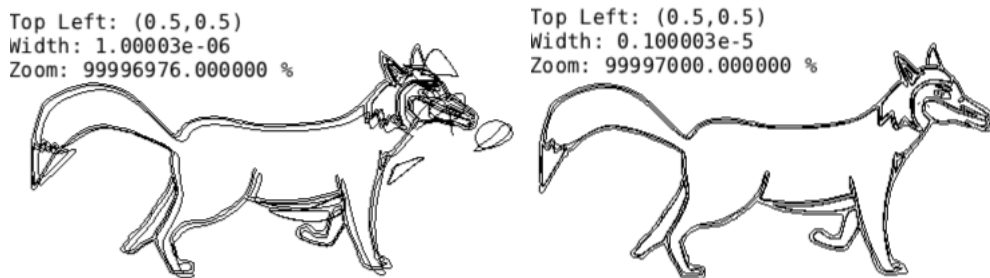


Figure 4.3: Effect of cumulative transformations applied to Paths

a) Path bounds represented using floats b) Path bounds represented using Rationals

4.2 Quantitative Measurements of Rendering Accuracy

A useful test SVG is a simple grid of horizontal and vertical lines separated by 1 pixel. When this SVG is correctly scaled to a view, all that should be visible is a coloured rectangle filling the screen. Increasing the magnification will reveal the grid of lines indicating how the original size of a pixel is scaled.

Figure 4.4 illustrates the effect of applying the view transformation (2.7) directly to the grid. When the grid is correctly rendered, as in Figure 4.4 a) it appears as a black rectangle. Further from the origin, not all pixels in the grid can be represented and individual lines become visible. As the distance from the origin increases, fewer pixel locations can be represented exactly after performing the view transformation.

An error of 1 ulp is increased by a factor of 10^6 to end up comparable to the size of the display ($0 \rightarrow 1$).

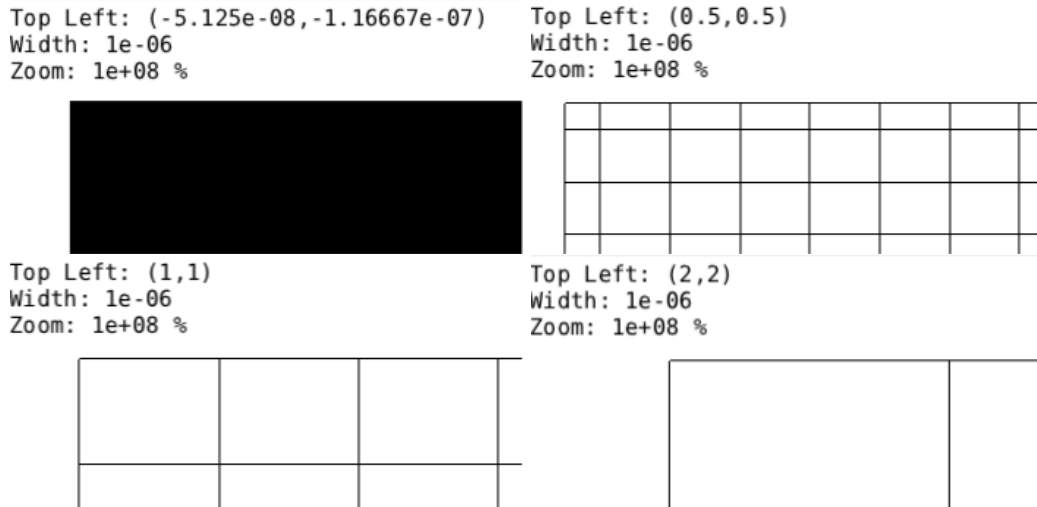


Figure 4.4: Effect of applying (2.7) to a grid of lines separated by 1 pixel
a) Near origin (denormals) b), c), d) Increasing the exponent of (v_x, v_y) by 1

4.2.1 Names of programs in figures

- single - Single precision IEEE-754 with (2.7) applied directly
- double - Double precision IEEE-754 with (2.7) applied directly
- cumul-single - Single precision IEEE-754 with cumulative transforms to Béziers
- cumul-double - Double precision IEEE-754 with cumulative transforms to Béziers
- path-single - Single precision IEEE-754 with cumulative transforms to Paths
- path-double - Single precision IEEE-754 with cumulative transforms to Paths
- path-rat - GNU MP Rationals with cumulative transforms to Paths

4.2.2 Precision for Fixed View

By counting the number of distinctly representable lines within a particular view, we can show the degradation of precision quantitatively. The test grid is added to each view rectangle.

Figure 4.5 shows how precision degrades with $(v_x, v_y) = (0.5, 0.5)$. A constant line at 1401 grid locations indicates no loss of precision.

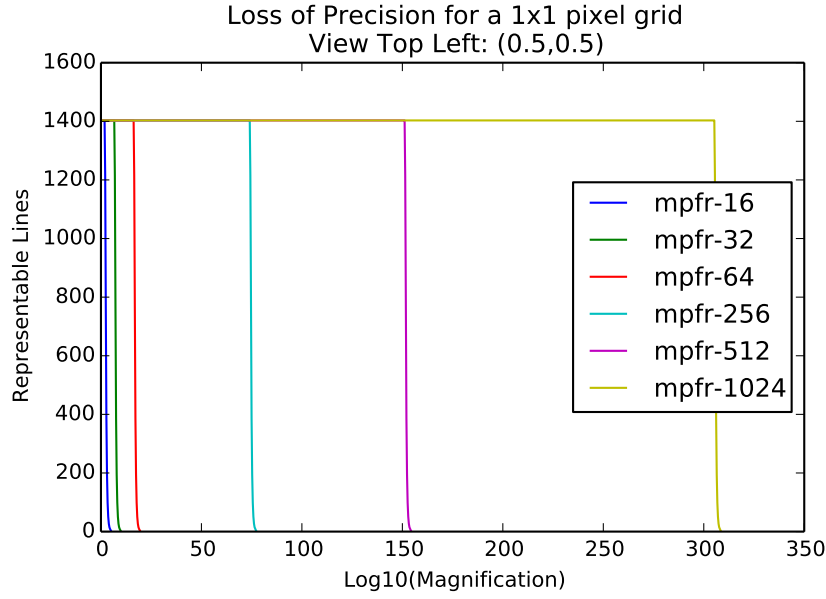


Figure 4.5: Loss of precision of the grid

4.2.3 Accumulated error after changing the View

Figure 4.6 shows the total error in the coordinates of each line in the grid after the view is scaled (zooming *out*) by repeated transformations. A constant line at 0 indicates no accumulated error.

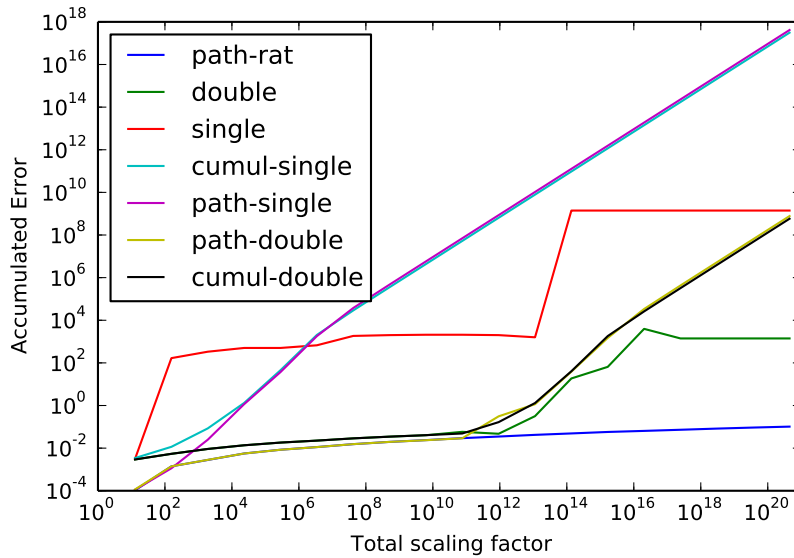


Figure 4.6: Error in the coordinates of the grid

By considering Figure 4.5 and 4.6, *path-rat* is the winner.

4.3 Performance Measurements whilst Rendering

As discussed above, we succeeded in preserving rendering accuracy as defined above for an arbitrary view. However this comes at a performance cost, as the size of the Rational number

representation must grow accordingly. Figures ?? and ?? were obtained by repeatedly clearing the document, scaling, and adding a fixed number of Bézier curves.

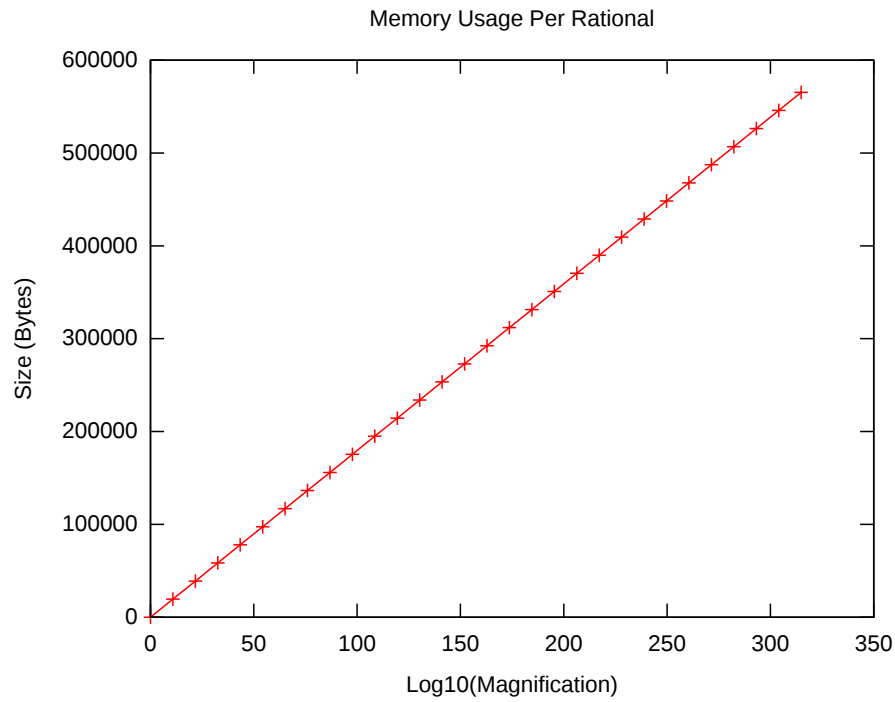


Figure 4.7: Memory used per Rational whilst zooming in

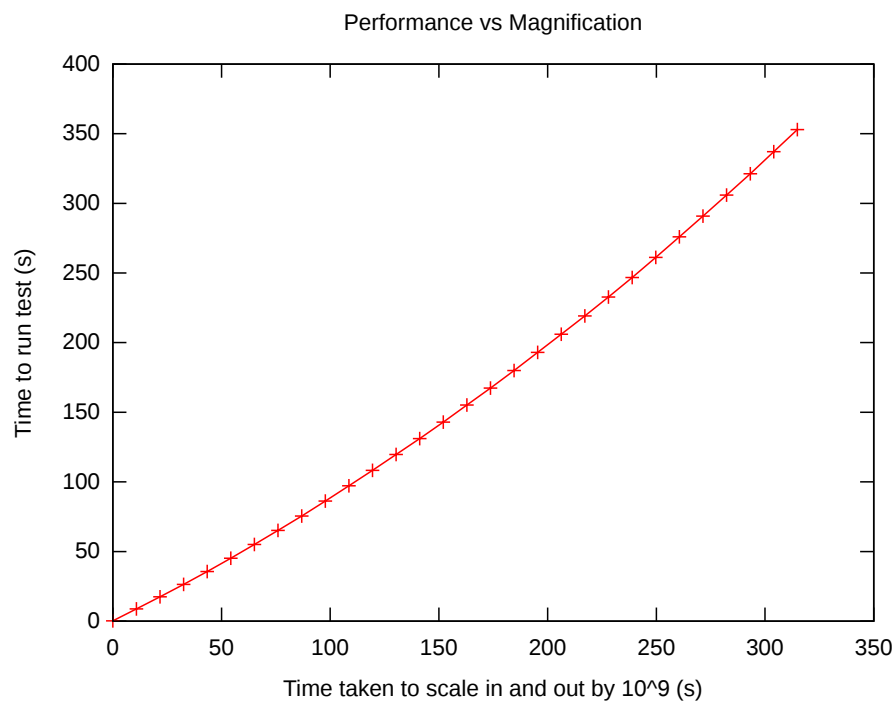


Figure 4.8: Time taken to scale about the fixed point

5. Conclusion

- What we have done?
 - Implemented a basic SVG viewer
 - Demonstrated how precision affects rendering vector graphics
 - Showed how the choice of transformations to apply affects rendering
 - Using GMP rationals, demonstrated the ability to render SVGs scaled to an arbitrary position in a document
- Possible future work
 - Implement more of the SVG standard (eg: Shading)
 - Trial alternative number representations, eg: MPFR with algorithm to set precision
 - Allow for saving and loading SVGs with arbitrary precision
 - Deal with zooming very far in to intersection of lines (requires subdividing paths)
 - Compare with David's Quadtree

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Note: We have collated most of these references at <http://szmoore.net/ipdf/documents/references/>