PHYSICAL REVIEW B VOLUME 2, NUMBER 4 15 AUGUST 1970

Theory of Optical Excitation of Plasmons in Metals*

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(Received 10 July 1969)

The usual Fresnel theory of reflection and transmission is shown to be incorrect when applied to materials in which longitudinal (electrostatic) polarization waves, such as the bulk plasma wave (plasmon), may propagate. A new macroscopic theory incorporating polarization waves is developed and applied to optical excitation of plasmons in metals. It is shown, that above ω_p ($\omega_p^2 = 4\pi ne^2/m$) the reflectance of a semi-infinite slab is diminished by plasmon excitation; and the mechanism of the plasma resonance absorption (predicted earlier in thin metal films) is discussed in detail. Numerical results for Na and K are presented along with a discussion of possible experiments.

I. INTRODUCTION

On the basis of Ferrell's physical picture predicting that plasma oscillations in metal foils would emit radiation,[1] Ferrell and Stern predicted that plasma oscillations in metal foils could be excited optically.[2] The following year in independent experiments, Yamaguchi,[3] and McAlister and Stern[4] observed anomalous absorption of polarized light by a thin Ag foil at the plasma frequency ω_p. McAlister and Stern[4] quantitatively accounted for this anomalous absorption by applying the Fresnel equations of reflection and transmission to a slab of material whose dielectric $\epsilon(\omega)$ corresponds to that of an electron plasma. But in contradiction to Ferrell's theory that only bulk-plasma oscillations couple to radiation,[1] this latter theory indicates that only surface-plasma oscillations are optically excited.

In this paper, we reexamine the theory of optical excitation of plasma oscillations, and show that bulk oscillations are indeed excited. The difficulty lies with the classical Fresnel equations, which are inadequate when applied to conducting media, i.e., media in which polarization waves are important. In Sec. II, we show why the classical equations are inadequate, examine such troublesome properties of conducting media as inhomogeneous waves and electromagnetic (EM) boundary conditions, and finally we derive the correct Fresnel equations which apply to conducting media. These new Fresnel equations are quite general, depending only on the dispersion relations for the EM (transverse) and the polarization (longitudinal) waves in the medium.

Although the dispersion relations for homogeneous waves in an unbounded plasma are well known, in Sec. III we rederive them for inhomogeneous waves, showing that the dispersion relations are essentially identical for the two cases.

In Sec. IV, we apply the Fresnel equations to semi-infinite slabs and thin films of potassium and sodium, and present some numerical results. Effects of the longitudinal bulk plasma wave on the reflectance, transmittance, and absorptance of EM waves are discussed along with possible experiments for their observation.

Section V relays the present theory to similar problems encountered in classical gas plasmas and suggests another application of the new theory.

II. FRESNEL EQUATIONS FOR CONDUCTING MEDIA

Until recently, the classical Fresnel equations of transmission and reflection of EM waves have been generally applied to any homogeneous isotropic media. In particular for conducting media, the real dielectric permittivity ϵ was replaced by a complex effective dielectric constant

$$\epsilon_{\text{eff}} = \epsilon + i4\pi\sigma/\omega\,, \tag{2.1}$$

where ω is the frequency of the wave and σ is the conductivity of the medium. Thus, for example, with this modification the classical equations have quite successfully been applied to the theory of metal optics.[5]

Since media in which electric polarization waves[6] can propagate may be considered conducting, the classical Fresnel equations were naturally extended to the problem of optical excitation of such polarization waves, e.g., plasma oscillations[4] or longitudinal optical phonons in polar crystals.[7] But these applications overlooked the fact that the classical equations consider only transverse EM waves with divergence-free electric fields, i.e., with $\vec{\nabla}\cdot\vec{E}=0$. This presents no difficulty in metal optics up to and including visible light frequencies, because in the metal there are no measurable bulk charge densities, and hence polarization fields, for frequencies less than the plasma frequency ω_p, which for most metals lies in the ultraviolet. But when polarization fields become important the classical Fresnel equations no longer apply. Two possible reasons why this fact has been overlooked are the ambiguity associated with inhomogeneous waves in conducting media and the incorrect treatment of boundary conditions for conducting media.[8]

A. Inhomogeneous Waves

By an inhomogeneous wave, we mean a wave whose surfaces of constant amplitude and constant phase do not coincide. To illustrate, consider a harmonic plane wave, represented by

$$\vec{E}(\vec{r},t)=\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} , \tag{2.2}$$

where $\vec{k}$ is a complex wave vector, with $\vec{k}=\vec{k}_1+i\vec{k}_2$. The physically measurable electric field represented by (2.2) is given by

$$\begin{aligned}\vec{E}(\vec{r},t)&=\mathrm{Re}[\vec{E}(r,t)]\\&=\mathrm{Re}[\vec{E}_0\exp(i\vec{k}_1\cdot\vec{r}-i\omega t)]\exp(-\vec{k}_2\cdot\vec{r}).\end{aligned} \tag{2.3}$$

Thus, at a given time, the surface of constant amplitude is $\vec{k}_2\cdot\vec{r}=$ constant, while the surface of constant phase is $\vec{k}_1\cdot\vec{r}=$ constant. Since we consider only wave vectors that are constant in space and time, these surfaces are planes normal to the direction of $\vec{k}_1$ and $\vec{k}_2$. From the above, we see that a wave is inhomogeneous whenever the real and imaginary parts of $\vec{k}$ have different directions. How an inhomogeneous wave can be produced will become apparent when we derive Snell's Law.

Snell's law. If $\vec{n}$ is a unit normal to the plane interface separating two media, let $\vec{n}\cdot\vec{r}=0$ define the surface of the interface. The existence of boundary conditions on the fields at any point on $\vec{n}\cdot\vec{r}=0$ at any time, requires that the space and time variation of all fields be the same on $\vec{n}\cdot\vec{r}=0$. Consequently, the phase factors in (2.2) for all the waves must be equal at $\vec{n}\cdot\vec{r}=0$ and independent of the nature of the boundary conditions. Since the time factors are trivially equal, the condition becomes

$$(\vec{k}_0\cdot\vec{r})_{\vec{n}\cdot\vec{r}=0}=(\vec{k}_j\cdot\vec{r})_{\vec{n}\cdot\vec{r}=0} , \tag{2.4}$$

where $\vec{k}_0$ is the wave vector of the incoming wave and $\vec{k}_j$ is the wave vector of any of the possible reflected or refracted waves. But

$$\vec{r}=(\vec{n}\cdot\vec{r})\vec{n}-\vec{n}\times(\vec{n}\times\vec{r}), \tag{2.5}$$

so (2.4) becomes

$$\vec{k}_0\cdot\vec{n}\times(\vec{n}\times\vec{r})=\vec{k}_j\cdot\vec{n}\times(\vec{n}\times\vec{r}) , \tag{2.6}$$

which by means of a vector identity can be written

$$(\vec{k}_0\times\vec{n}-\vec{k}_j\times\vec{n})\cdot(\vec{n}\times\vec{r})=0. \tag{2.7}$$

Thus, Snell's law, as expressed by (2.7), states that $\vec{n}$ and separately the real and imaginary parts of the wave vectors $\vec{k}$ are coplanar, and the components of the wave vectors parallel to the interface are equal.

Let us now consider a homogeneous wave in vacuum incident with an angle θ_0 on a semi-infinite medium in which the wave vector is complex, as in Fig. 1. Snell's law requires

$$\begin{aligned}k_0\sin\theta_0&=k_1\sin\theta ,\\ 0&=k_2\sin\varphi ,\end{aligned} \tag{2.8}$$

and we see how and why obliquely incident homogeneous waves produce inhomogeneous waves in lossy media such as metals.

We see, therefore, that while the direction of $\vec{E}$ in a homogeneous EM wave is well defined as being transverse to the direction of propagation, care must be taken in describing waves in lossy media. For example, the statement that $\vec{E}$ of an EM wave obliquely excited in a metal has a longitudinal component[9] should be taken to mean that the physical field has a component in the direction

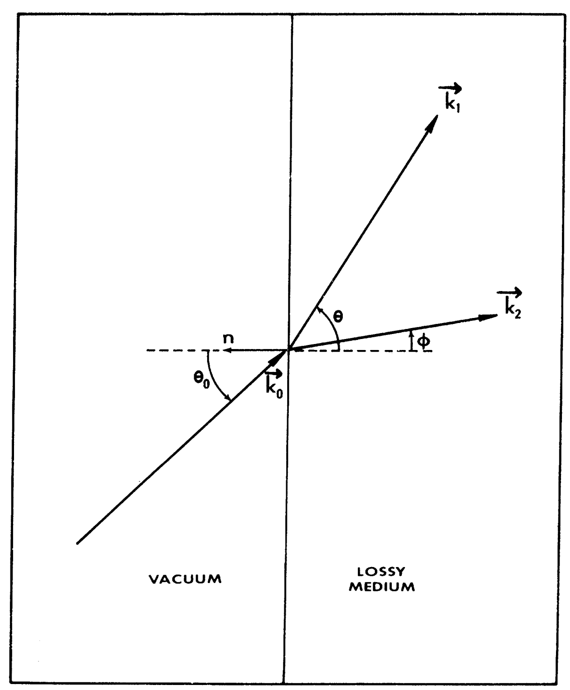

FIG. 1. Wave $\vec{k}_0$ incident on some lossy medium excites an inhomogeneous wave $\vec{k}=\vec{k}_1+\vec{k}_2$.

of phase propagation, but it does not mean that $\vec{\nabla}\cdot\vec{E}\neq 0$.

B. Boundary Conditions

On a boundary between two media, on which there are no external charges or currents, the usual macroscopic boundary conditions are the continuity of the tangential electric field $\vec{E}$, the normal displacement vector $\vec{D}$, the tangential magnetic field $\vec{H}$, and the normal magnetic flux density $\vec{B}$.[10] These boundary conditions follow from the standard arguments which apply Maxwell's equations to pillboxes and infinitesimal circuits passing through the boundary surface. If one or both of the media are conducting, the usual procedure has been to incorporate the current $\vec{J}$ into an effective $\vec{D}$ by means of relation (2.1). But recently this procedure has come under criticism.[11] In the following paragraphs, we will carefully examine the derivation of the boundary conditions and determine the correct set for a conducting medium.

Maxwell's equation for the magnetic field may be written

$$\vec{\nabla}\times\vec{H}=\frac{1}{c}\frac{\partial}{\partial t}\vec{E}+\frac{4\pi}{c}(\vec{J}_{\text{pol}}+\vec{J}_{\text{cond}})=\frac{4\pi}{c}\vec{J}_{\text{total}}, \quad (2.9)$$

i.e., the total current consists of three components: the pure displacement current (first term), the polarization current, and the conduction current. The distinction between the displacement current and the other currents is obvious since the former is independent of any charge. In regard to the other two, Maxwell considered that the polarization current differs from the conduction current in that the latter constitutes the motion of free charges, while the former is associated with bound charges. But this distinction can strictly be made only for static fields and becomes meaningless for time-dependent fields, especially those of high frequency. Thus, from the microscopic point of view, the current density in a medium is the sum of the polarization and conduction currents. So if[12]

$$\vec{J}=-\vec{\alpha}\cdot\frac{\partial}{\partial t}\vec{E}+\vec{\sigma}\cdot\vec{E}, \quad (2.10)$$

where $\vec{\alpha}$ and $\vec{\sigma}$ are the real polarizability and conductivity tensors, respectively,[13] the current can be written in the form of a polarization current by replacing $\vec{\alpha}$ by a complex polarizability

$$\vec{\alpha}'=\vec{\alpha}+i\vec{\sigma}/\omega, \quad (2.11)$$

or, conversely, in the form of a conduction current by replacing $\vec{\sigma}$ by a complex conductivity

$$\vec{\sigma}'=\vec{\sigma}-i\omega\vec{\alpha}. \quad (2.12)$$

If we now take the divergence of Eq. (2.9) and apply it to the usual pillbox argument, we find that the normal component of the total current (displacement and charge current) is continuous across the boundary surface. Let us note that for the case of harmonic time dependence, $\vec{J}_{\text{total}}$ and $\vec{D}_{\text{eff}}$ are related by a constant $4\pi i/\omega$, and the above condition becomes equivalent to the continuity of normal $\vec{D}_{\text{eff}}$.

Next, we apply the pillbox argument to the equation of continuity. From our microscopic viewpoint all charges, which we treat explicitly, are bulk charges, in the limit that the volume vanishes the enclosed charge density vanishes. Hence, we find the normal component of the charge current density is continuous.

From this condition and the previous one, it follows that the normal component of the displacement current is also continuous. Because the condition that tangential $\vec{H}$ is continuous comes about from applying Eq. (2.9) to a circuit argument, it is fairly obvious that the conditions on tangential $\vec{H}$ and normal $\vec{J}_{\text{total}}$ are equivalent. Thus, of the four conditions on normal $\vec{J}_{\text{total}}$, $\vec{J}_{\text{charge}}$, and $1/4\pi(\partial/\partial t)\vec{E}$ and tangential $\vec{H}$, only two are independent, which, for reasons of convenience, we chose to be tangential $\vec{H}$ and normal displacement current.

The last two conditions, the continuity of tangential $\vec{E}$ and normal $\vec{H}$ (or $\vec{B}$) follow as usual, but again, since the two are related only one is necessary. In summary, our set of boundary conditions are the continuity of tangential $\vec{E}$ and $\vec{H}$ and normal displacement current.[14]

C. Fresnel Equations for a Single Surface

Consider a plane linearly polarized EM wave incident on a semi-infinite medium at an angle θ_0, as in Fig. 2. Let $\vec{s}$, $\vec{n}$ and $\vec{p}$ be a triplet of orthogonal unit vectors, $\vec{n}$ being normal to the surface separating the two media and $\vec{p}$ in the plane of incidence. For simplicity, we assume the medium on the left to be a vacuum or at worst a dispersionless dielectric, i.e., incapable of supporting polarization waves, while the medium on the right is some general conducting, but nonmagnetic, medium. Since by assumption only transverse EM waves can propagate in medium 0 (half-space on the left), we have only the incident and reflected EM wave with wave vectors $\vec{k}_0$ and $\vec{k}_R$, whose magnitudes are equal. But in the conducting medium we, in general, expect besides the EM wave field a polarization field. We assume, and in the next section will partially justify, that these fields may be separated into two noninteracting waves: a transverse EM wave with divergence-free fields

$$\vec{k}_T\cdot\vec{E}_T=0, \quad (2.13)$$

and a longitudinal polarization wave with irrotational fields

$$\vec{k}_L \times \vec{E}_L = 0. \tag{2.14}$$

The behavior of the waves in the medium is determined by the electromagnetic properties of the medium, expressed by the dispersion relations $k(\omega)$ which determine the magnitudes of the wave vectors, and by Snell's law which determines the direction of the wave vectors,

$$\vec{k}_0 \cdot \vec{p} = \vec{k}_R \cdot \vec{p} = \vec{k}_T \cdot \vec{p} = \vec{k}_L \cdot \vec{p}\,,$$
$$-\vec{k}_0 \cdot \vec{n} = \vec{k}_R \cdot \vec{n}. \tag{2.15}$$

Since by superposition, an arbitrary direction of polarization can be resolved into two cases, one with $\vec{E}$ in the plane of incidence (p polarized) and the other with $\vec{E}$ normal to the plane of incidence (s polarized), we will consider these polarizations separately.

p polarization. For the p-polarized case, let the incident, reflected, and transmitted waves be

$$\vec{E}_0(\vec{r}, t) = (\vec{s} \times \vec{K}_0) E_0 \exp(i\vec{k}_0 \cdot \vec{r} - i\omega t), \tag{2.16a}$$
$$\vec{E}_R(\vec{r}, t) = (\vec{s} \times \vec{K}_R) E_0 R_P \exp(i\vec{k}_R \cdot \vec{r} - i\omega t), \tag{2.16b}$$
$$\vec{E}_T(\vec{r}, t) = (\vec{s} \times \vec{K}_T) E_0 T_P \exp(i\vec{k}_T \cdot \vec{r} - i\omega t), \tag{2.16c}$$
$$\vec{E}_L(\vec{r}, t) = \vec{K}_L E_0 L_P \exp(i\vec{k}_L \cdot \vec{r} - i\omega t), \tag{2.16d}$$

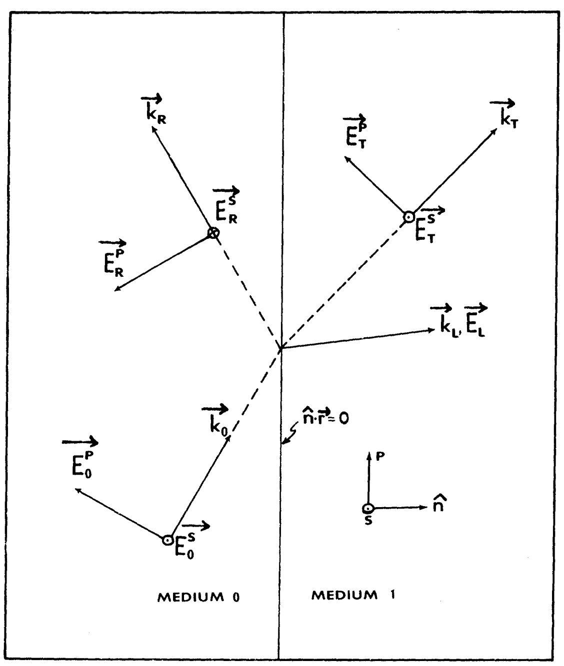

FIG. 2. Reflection and refraction of s- and p-polarized waves at a boundary between a nonconducting (0) and conducting (1) medium.

where $\vec{K}$ is the normalized wave vector

$$\vec{K}_\alpha = c\vec{k}_\alpha / \omega. \tag{2.17}$$

Note the notation we have chosen shows explicitly that the EM fields are divergence-free and the polarization fields are irrotational. Since $\vec{E}$ and $\vec{H}$ are related by

$$\vec{H} = \vec{K} \times \vec{E}, \tag{2.18}$$

the corresponding magnetic fields are

$$\vec{H}_0(\vec{r}, t) = \vec{s}\,\epsilon_0 E_0 \exp(i\vec{k}_0 \cdot \vec{r} - i\omega t), \tag{2.19a}$$
$$\vec{H}_R(\vec{r}, t) = \vec{s}\,\epsilon_0 E_0 R_P \exp(i\vec{k}_R \cdot \vec{r} - i\omega t), \tag{2.19b}$$
$$\vec{H}_T(\vec{r}, t) = \vec{s}\,\epsilon_T E_0 T_P \exp(i\vec{k}_T \cdot \vec{r} - i\omega t), \tag{2.19c}$$

where $\epsilon = \vec{K} \cdot \vec{K}.$ (2.20)

Later, when discussing dispersion relations, we shall see that ϵ corresponds to the transverse dielectric of the media.

Applying the condition of continuity of tangential $\vec{E}$,

$$\vec{n} \times (\vec{E}_0 + \vec{E}_R)_{\hat{n}\cdot\vec{r}=0} = \vec{n} \times (\vec{E}_T + \vec{E}_L)_{\hat{n}\cdot\vec{r}} = 0, \tag{2.21}$$

to (2.16), we find, after some algebra,

$$(\vec{K}_0)_n [1 - R_P] = (\vec{K}_T)_n T_P + (\vec{K})_P L_P\,, \tag{2.22}$$

where the subscripts on the $\vec{K}$'s indicate the components, e.g., $(\vec{K})_n = \vec{n} \cdot \vec{K}$. Similarly, the continuity of the magnetic fields yields

$$\epsilon_0 [1 + R_P] = \epsilon_T T_P. \tag{2.23}$$

Finally, the continuity of the normal displacement current, i.e., E, at the boundary yields

$$(\vec{K})_P [1 + R_P] = (\vec{K})_P T_P - (\vec{K}_L)_n L_P. \tag{2.24}$$

From Eqs. (2.22)–(2.24) we obtain the p-polarization Fresnel equations for a conducting half-space

$$R_P = (\alpha - \beta - \gamma)/(\alpha + \beta + \gamma), \tag{2.25a}$$
$$T_P = 2\,\epsilon_0 (\vec{K}_0)_n / (\alpha + \beta + \gamma), \tag{2.25b}$$
$$L_P = [2\gamma\,(\vec{K}_0)_n / (\vec{K})_P] / (\alpha + \beta + \gamma), \tag{2.25c}$$

where
$$\alpha = \epsilon_T (\vec{K}_0)_n\,, \tag{2.26a}$$
$$\beta = \epsilon_0 (\vec{K}_T)_n\,, \tag{2.26b}$$
$$\gamma = (\vec{K})_P^2 [\epsilon_0 - \epsilon_T] / (\vec{K}_L)_n. \tag{2.26c}$$

For comparison, the corresponding classical Fresnel equations are

$$R_P^c = (\alpha - \beta)/(\alpha + \beta), \tag{2.27a}$$
$$T_P^c = 2\,\epsilon_0 (\vec{K}_0)_n / (\alpha + \beta). \tag{2.27b}$$

The above are easily obtained by neglecting the polarization wave and applying the continuity conditions to tangential $\vec{E}$ and $\vec{H}$.[15]

s polarization. For the s-polarized case no $\vec{E}$ lies in the plane of propagation so no polarization wave is excited. Using the continuity of tangential $\vec{E}$ and $\vec{H}$, it is easy to show that the Fresnel equations are identical to the classical equations

$$R_s = [(\vec{K}_0)_n - (\vec{K}_T)_n]/[(\vec{K}_0)_n + (\vec{K}_T)_n], \qquad (2.28a)$$

$$T_s = 2(\vec{K}_0)_n/[(\vec{K}_0)_n + (K_T)_n]. \qquad (2.28b)$$

This is not surprising since polarization waves play no role in this case. But if the medium were magnetic, i.e., if "magnetic polarization" currents could be excited, the classical equations for s polarization would also be incorrect.

Since the s-polarization Fresnel equations for nonmagnetic media are independent of polarization waves, we will henceforth consider only the p-polarization equations. Comparing the new equations[16] (2.25) with the classical equations (2.27), we see they become identical whenever γ vanishes. This quantity is proportional to three factors: the sine squared of the angle of incidence $K_P^2 = \epsilon_0 \sin^2\theta_0$; the difference between the transverse dielectrics of the two media $\epsilon_0 - \epsilon_T$; and the normal component of the longitudinal-polarization-wave wavelength in units of the vacuum wavelength $(\vec{K}_L)_n^{-1}$. This last factor is the important one. If no polarization wave can propagate, i.e., if the wave is exponentially damped in a short distance, $(\vec{K}_L)_n^{-1}$ is small and imaginary, and hence γ is negligible, and the results are identical to those given by the classical equations.[17] In other words, the factor γ, and therefore the new Fresnel equations, become important only when polarization waves propagate with finite wavelengths.

Wave energies and their conservation. Because observation are usually made on the reflected or transmitted intensities or energies rather than amplitudes, it is useful to relate the Fresnel equations to the corresponding coefficients of energy flux. Since the time-averaged energy flow in an EM wave is given by the real part of the complex Poynting vector

$$\vec{S} = (c/4\pi)\tfrac{1}{2}\,\mathrm{Re}[\vec{E}\times\vec{H}^*], \qquad (2.29)$$

the energy fluxes of the incident, reflected, and transmitted EM waves are

$$\vec{S}_0 = (c/8\pi)\vec{K}_0\epsilon_0 E_0^2, \qquad (2.30a)$$

$$\vec{S}_R = (c/8\pi)\vec{K}_R\epsilon_0 |R_P|^2 E_0^2, \qquad (2.30b)$$

$$\vec{S}_T = (c/8\pi)\,\mathrm{Re}[\vec{K}_T\epsilon_T^*]\,|T_P|^2 E_0^2 \exp(-2\vec{k}_{T_2}\cdot\vec{r}). \qquad (2.30c)$$

The polarization wave, however, has no Poynting vector associated with it, transmitting energy mechanically by the motion of the charges.

The transmittance and reflectance of EM waves, i.e., the fractions of incident energy transmitted and reflected, are defined as

$$\vec{T} = (\vec{n}\cdot\vec{S}_T/\vec{n}\cdot\vec{S}_0)_{\vec{n}\cdot\vec{r}=0}, \qquad (2.31a)$$

$$\vec{R} = -(\vec{n}\cdot\vec{S}_R/\vec{n}\cdot\vec{S}_0)_{\vec{n}\cdot\vec{r}=0}. \qquad (2.31b)$$

From Eqs. (2.30), the expressions for $\vec{T}$ and $\vec{R}$ are

$$\vec{T} = \mathrm{Re}[\epsilon_T^*(\vec{K}_T)_n/(\vec{K}_0)_n]\,|T_P|^2, \qquad (2.32a)$$

$$\vec{T} = 4\,\mathrm{Re}[\alpha^*\beta]/|\alpha+\beta+\gamma|^2, \qquad (2.32b)$$

$$\vec{R} = |R_P|^2 = |(\alpha-\beta-\gamma)/(\alpha+\beta+\gamma)|^2 \qquad (2.33)$$

Applying Poyntings theorem to the energy flow across the surface $\vec{n}\cdot\vec{r}=0$, we find

$$\vec{I} - \vec{R} = \vec{T} + \vec{T}', \qquad (2.34)$$

with
$$\vec{T}' = 4\,\mathrm{Re}[\alpha^*\gamma]/|\alpha+\beta+\gamma|^2, \qquad (2.35a)$$

$$\vec{T}' = \mathrm{Re}[\epsilon_T^* L_P T_P^* (\vec{K})_P/(\vec{K}_0)_n]. \qquad (2.35b)$$

Thus, it appears that $\vec{T}'$ is the fraction of the incident energy flowing into the polarization wave.

D. Fresnel Equations for a Slab

Some of the most interesting and useful phenomena in optics are produced by the interference of EM waves in thin dielectric slabs. Because we expect that polarization waves may exhibit similar interference properties, we next calculate the reflection and transmission equations for a slab of material capable of supporting polarization waves.

Suppose our conducting medium is bounded by two planes, one being $\vec{n}\cdot\vec{r}=0$ and the other being $\vec{n}\cdot\vec{r}=d$, as in Fig. 3. To the left of this slab of thickness d is a dispersionless dielectric ϵ_0, and to the right is another dispersionless dielectric ϵ_2. Since no polarization waves can be excited for s-polarized incidence we will consider only the

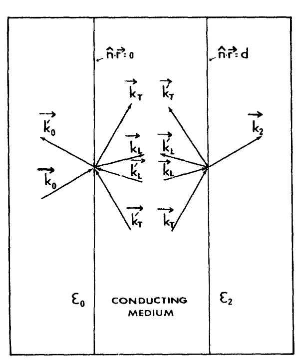

FIG. 3. Reflection and transmission by a conducting slab of thickness d.

p-polarized case. The incident and reflected waves in ϵ_0 are

$$\vec{E}_0 = (\vec{s}\times\vec{K}_0)\exp(i\vec{k}_0\cdot\vec{r}-i\omega t) \quad , \tag{2.36a}$$

$$\vec{H}_0 = \vec{s}\epsilon_0\exp(i\vec{k}_0\cdot\vec{r}-i\omega t) \quad , \tag{2.36b}$$

$$\vec{E}_0' = (\vec{s}\times\vec{K}_0')R\exp(i\vec{k}_0'\cdot\vec{r}-i\omega t) \quad , \tag{2.36c}$$

$$H_0' = \vec{s}\epsilon_0\exp(i\vec{k}_0'\cdot\vec{r}-i\omega t) \quad , \tag{2.36d}$$

where the primed quantities refer to the reflected waves. The conducting medium contains four waves, polarization and EM waves propagating to the right and to the left:

$$\vec{E}_T = (\vec{s}\times\vec{K}_T)A_T\exp(i\vec{k}_T\cdot\vec{r}-i\omega t) \quad , \tag{2.37a}$$

$$\vec{H}_T = \vec{s}\epsilon_T A_T\exp(i\vec{k}_T\cdot\vec{r}-i\omega t) \quad , \tag{2.37b}$$

$$\vec{E}_T' = (\vec{s}\times\vec{K}_T')A_T'\exp(i\vec{k}_T'\cdot\vec{r}-i\omega t) \quad , \tag{2.37c}$$

$$\vec{H}_T' = \vec{s}\epsilon_T A_T'\exp(i\vec{k}_T'\cdot\vec{r}-i\omega t) \quad , \tag{2.37d}$$

$$\vec{E}_L = \vec{K}_L A_L\exp(i\vec{k}_L\cdot\vec{r}-i\omega t) \quad , \tag{2.37e}$$

$$\vec{E}_L' = \vec{K}_L' A_L'\exp(i\vec{k}_L'\cdot\vec{r}-i\omega t) \quad . \tag{2.37f}$$

Finally, the transmitted wave in ϵ_2 is

$$\vec{E}_2 = (\vec{s}\times\vec{K}_2)T\exp(i\vec{k}_2\cdot\vec{r}-i\omega t) \quad , \tag{2.38a}$$

$$\vec{H}_2 = \vec{s}\epsilon_2 T\exp(i\vec{k}_2\cdot\vec{r}-i\omega t) \quad . \tag{2.38b}$$

Snell's law requires the p components of all wave vectors to be equal (law of refraction), and the n components of the primed wave vectors to be equal to the negative n component of the corresponding unprimed wave vector (law of reflection). Applying the continuity conditions and Snell's law to $\vec{E}$ and $\vec{H}$ at the surface $\vec{n}\cdot\vec{r}=0$ we obtain

$$(\vec{K}_0)_n[1-R] = (\vec{K}_T)_n[A_T - A_T'] + (\vec{K})_p[A_L + A_L'] \quad , \tag{2.39a}$$

$$(\vec{K})_p[1+R] = (\vec{K})_p[A_T + A_T'] - (\vec{K}_L)_n[A_L - A_L'] \quad , \tag{2.39b}$$

$$\epsilon_0[1+R] = \epsilon_T[A_T + A_T'] \quad . \tag{2.39c}$$

Similarly, at the surface $\vec{n}\cdot\vec{r}=d$, we have

$$(\vec{K}_T)_n[A_T\varphi_T - A_T'\varphi_T'] + (K)_p[A_L\varphi_L + A_L'\varphi_L'] = (\vec{K}_2)_n T\varphi_2 \quad , \tag{2.40a}$$

$$(\vec{K})_p[A_T\varphi_T + A_T'\varphi_T'] - (\vec{K}_L)_n[A_L\varphi_L - A_L'\varphi_L'] = (\vec{K})_p T\varphi_2 \quad , \tag{2.40b}$$

$$\epsilon_T[A_T\varphi_T + A_T'\varphi_T'] = \epsilon_2 T\varphi_2 \quad , \tag{2.40c}$$

where
$$\varphi_\alpha = \exp(i\vec{n}\cdot\vec{k}_\alpha d) \quad , \tag{2.41a}$$

$$\varphi'\varphi = 1 \quad . \tag{2.41b}$$

Solving (2.39) and (2.40) for R and T, we find the Fresnel equations of reflection and transmission for a slab to be

$$R = X/Z \ , \tag{2.42a}$$

$$T\varphi_2 = Y/Z \quad , \tag{2.42b}$$

where
$$X = (1-\varphi_T\varphi_L)(A_0D_2 - D_0A_2\varphi_T\varphi_L) - (\varphi_T-\varphi_L)(C_0B_2\varphi_T - B_0C_2\varphi_L) \ , \tag{2.43a}$$

$$Y = (A_0+D_0)\varphi_T\varphi_L[(\varphi_T - \varphi_T')(A_0 - B_0) + (\varphi_L - \varphi_L')(A_0 - C_0)] \quad , \tag{2.43b}$$

$$Z = (1-\varphi_T\varphi_L)(D_0D_2 - A_0A_2\varphi_T\varphi_L) - (\varphi_T-\varphi_L)(B_0B_2\varphi_T - C_0C_2\varphi_L) \ , \tag{2.43c}$$

and
$$A = \alpha - \beta - \gamma \quad , \tag{2.44a}$$

$$B = \alpha - \beta + \gamma \quad , \tag{2.44b}$$

$$C = \alpha + \beta - \gamma \quad , \tag{2.44c}$$

$$D = \alpha + \beta + \gamma \quad . \tag{2.44d}$$

The subscript 2 on (2.43) indicates that ϵ_0 and $\vec{K}_0$ are replaced by ϵ_2 and $\vec{K}_2$ in corresponding expressions for α, β, and γ. Although the above expressions are complicated, the appearance of the phase factor φ_L, as well as φ_T, indicates interference by polarization waves which we will discuss shortly.

As in the case of a single boundary, the features due to polarization waves are best seen by comparing the above Fresnel equations with the corresponding classical equations. They are not difficult to derive, and because they can be found in the literature[18] we will write them directly,

$$R^c = (A_0^c D_2^c - D_0^c A_2^c\varphi_T^2)/(D_0^c D_2^c - A_0^c A_2^c\varphi_T^2), \tag{2.45a}$$

$$T^c = [(D_0^c)^2 - (A_0^c)^2]\varphi_T/(D_0^c D_2^c - A_0^c A_2^c\varphi_T^2), \tag{2.45b}$$

with
$$A^c = \alpha - \beta \quad , \tag{2.46a}$$

$$D^c = \alpha + \beta \quad , \tag{2.46b}$$

and the subscripts having the same meaning as before. Again the factor which makes (2.42)–(2.43) differ from (2.45) is $(\vec{K}_L)_n$. If no polarization wave exists, $\mathrm{Im}(\vec{K}_L)_n \to \infty$, then φ_L, $\gamma \to 0$, and the new equations reduce to the classical. Note that in the limit $d\to\infty$, the equations for a slab reduce to the single-boundary case, provided the EM and polarization waves are at least slightly damped.

As was done in the single-boundary case, it is useful to relate Eqs. (2.42) to the transmittance and reflectance. It is an easy matter to show that they are

$$R = |X/Z|^2 \quad , \tag{2.47a}$$

$$T = |Y/Z|^2 \quad , \tag{2.47b}$$

and since energy that is neither reflected nor transmitted must be absorbed, the absorptance is

$$A = 1 - R - T \quad . \tag{2.47c}$$

Multiple-reflection resonance. If we take the classical Fresnel equations (2.45), and for simplicity assume the media on either side of the slab are identical and the EM waves are undamped, the classical reflectance and transmittance for a slab may be written

$$R^c = \frac{2\vec{R}^c(1-\cos 2\psi_T)}{(1-\vec{R}^c)^2 + 2\vec{R}^c(1-\cos 2\psi_T)} , \tag{2.48a}$$

$$T^c = \frac{(1-\vec{R}^c)^2}{(1-\vec{R}^c)^2 + 2\vec{R}^c(1-\cos 2\psi_T)} , \tag{2.48b}$$

where $\vec{R}^c$ is the reflectance at a single surface,

$$\vec{R}^c = |(\alpha-\beta)/(\alpha+\beta)|^2 \tag{2.49}$$

and $$\psi_T = (\vec{K}_T)_n 2\pi d/\lambda . \tag{2.50}$$

Equations (2.48) demonstrate the usual multiple-reflection resonance in thin films that have applications, for example, in interference filters.

In analogy, let us look for multiple reflection of the polarization wave and see under what conditions it may be observed. In general, Eqs. (2.42) indicate a complex structure that involves multiple reflection of both EM and polarization waves. So observation of the polarization wave will be difficult unless the EM wave can be neglected. One possibility is the case where the wavelength of the EM wave λ_T is many orders of magnitude larger than the wavelength of the polarization wave λ_L. For this case, we choose the slab thickness to be

$$\lambda_T \gg d > \lambda_L , \tag{2.51}$$

so that we may neglect ψ_T and let $\varphi_T \to 1$. Again assuming the same media on each side of the slab and undamped waves, so that α, β, and γ are real, we find

$$R = \frac{\gamma^2(1-\cos\psi_L)}{\alpha^2(1+\cos\psi_L)+\gamma^2(1-\cos\psi_L)} , \tag{2.52a}$$

$$T = \frac{\alpha^2(1+\cos\psi_L)}{\alpha^2(1+\cos^2\psi_L)+\gamma^2(1-\cos\psi_L)} , \tag{2.52b}$$

where $$\psi_L = (\vec{K}_L)_n 2\pi d/\lambda . \tag{2.53}$$

What is of particular significance about Eqs. (2.52) is the fact that even when $\alpha \gg \gamma$, structure in T or R will be observed when

$$\begin{aligned} &\psi_L = n\pi , \quad n \text{ odd} \\ \text{or} \quad &n(\tfrac{1}{2}\lambda_L)_n = d , \quad n \text{ odd} \end{aligned} \tag{2.54}$$

i.e., when the thickness is equal to an odd number of half-wavelengths of the polarization wave. Because, in general, $\alpha \gg \gamma$ this property of Eqs. (2.52) is more important for the observation of bulk plasma waves of finite wavelength.

III. DISPERSION RELATIONS FOR INHOMOGENEOUS WAVES IN AN ELECTRON GAS

A dispersion relation, in the sense that we use the term, gives the wave vector k (which, in general, may be complex) as a function of the frequency ω (which we assume to be real). Dispersion relations for waves in an electron plasma have been calculated by many authors and can be found in most textbooks on plasma physics. But in Sec. II., we saw that waves in lossy media are, in general, inhomogeneous, while all previous calculations have been for homogeneous waves. For this reason, and to show that even inhomogeneous waves naturally separate into divergence-free (EM) and curl-free (polarization) waves, the dispersion relations for inhomogeneous waves in a free-electron model of a metal will be calculated below.

The motion of the distribution function f of the electrons in a uniform neutralizing positive background is governed by the Boltzmann transport equation

$$\frac{\partial}{\partial t} f + \vec{v}\cdot\vec{\nabla}_r f + \frac{\vec{F}}{m}\cdot\vec{\nabla}_v f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} . \tag{3.1}$$

Making the Vlasov approximation, where the EM interaction among electrons is replaced by a self-consistent field which is incorporated into $\vec{F}$, and assuming the electron gas is disturbed from equilibrium by an EM force field proportional to $\exp(i\vec{k}\cdot\vec{r} - i\omega t)$, $\vec{F}$ may be written as

$$\vec{F} = -e[\vec{E} + (1/c)\vec{v}\times\vec{B}] , \tag{3.2a}$$

$$\vec{F} = -e[\vec{k}\vec{v}/\omega + (1+\vec{k}\cdot\vec{v}/\omega)\vec{I}]\cdot\vec{E} . \tag{3.2b}$$

The last line arises because the self-consistent as well as the EM fields must satisfy Maxwell's equations.

Collision term. The collision term which involves electron scattering by impurities, phonons, etc., will be treated by the relaxation-time ansatz

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = -\tau^{-1}[f(\vec{r},\vec{v},t) - f_s(\vec{r},\vec{v},t)] , \tag{3.3}$$

where τ^{-1} is the collision frequency, f is the distribution before scattering, and f_s is the distribution after scattering, i.e., the *local equilibrium* distribution. The total equilibrium distribution for the electrons is

$$f_0 = (3N_0/4\pi\epsilon_F^{3/2})\mathfrak{M}^{3/2}\{1+\exp[(\epsilon-\epsilon_F)/kT]\}^{-1}, \tag{3.4}$$

where N_0 is the average density, ϵ the energy per particle, ϵ_F the Fermi energy, and kT the thermal energy. Because the Fermi energy is a function of density, at $T=0$ being

$$\epsilon_F^0 = (\hbar^2/2m)(3\pi^2 N)^{2/3} , \tag{3.5}$$

and because the density varies in space and time (due to the disturbance) as

$$N(\vec{r},t) = N_0 + N_1 e^{i(\vec{k}\cdot\vec{r}-\omega t)} , \tag{3.6}$$

the local equilibrium distribution also varies. Assuming the density variations are small compared to the equilibrium value, i.e., $N_0 \gg N_1$, we can make a linear expansion of f_s about f_0,

$$f_s = f_0 + \left(\frac{\partial f}{\partial \epsilon_F}\right)\left(\frac{\partial \epsilon_F}{\partial N}\right) N_1 . \tag{3.7}$$

Starting with Eq. (3.7), quantities with subscript 1 are small compared with the equilibrium value, subscript 0, and have space-time dependence $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, which will not be written explicitly.

Before proceeding any further, something must be said about the temperature of the electrons and their equilibrium distribution. In general, (3.4) is difficult to handle except in the limit of very low or very high temperatures. But because the Fermi energy of electrons at metallic densities is so much greater than the thermal energy at room temperature, (3.4) may be approximated to be at $T=0$. On the other hand, at semiconductor densities, (3.4) may be approximated to be at a high temperature, i.e., by the Maxwell-Boltzmann distribution function. Since our interest is primarily in metallic plasmas, we will consider only the former case, stating that the high-temperature results are analogous. In the low-temperature approximation, (3.7) becomes

$$f_s = f_0 - \left(\frac{\partial f_0}{\partial \epsilon}\right) \frac{2}{3}\frac{\epsilon_F}{N_0} N_1 , \tag{3.8}$$

which may be substituted into (3.1) along with (3.2) and (3.3). Upon assuming a linear solution of the form

$$f(\vec{r}, \vec{v}, t) = f_0 + f_1 , \tag{3.9}$$

the Boltzmann equation (3.1) has the solution

$$f_1 = e\left(\frac{\partial f_0}{\partial \epsilon}\right) \frac{\vec{v}\cdot\vec{E}_1}{\tau^{-1} - i\omega - i\vec{k}\cdot\vec{v}} - \frac{2}{3}\frac{\epsilon_F}{N_0}\left(\frac{\partial f_0}{\partial \epsilon}\right) \times \frac{\tau^{-1} N_1}{\tau^{-1} - i\omega - i\vec{k}\cdot\vec{v}} . \tag{3.10}$$

A. Conductivity Tensor

From the distribution function, we obtain the current density through the moment equation

$$\vec{J}(\vec{r}, \vec{t}) = -e \int d^3v\, \vec{v} f(\vec{r}, \vec{v}, t) . \tag{3.11}$$

Since f_0 is symmetric in $\vec{v}$, only f_1 contributes to the current; this contribution has two parts: The conductivity current proportional to $\vec{E}_1$ and the diffusion current proportional to N_1. The latter is due to the space and time variation of the local equilibrium density and is proportional to the collision frequency. Thus,

$$\vec{J}_1 = \vec{\sigma}'(\vec{k}, \omega)\cdot\vec{E}_1 - e\omega\vec{D}(\vec{k}, \omega) N_1 , \tag{3.12}$$

where
$$\vec{\sigma}' = e^2 \int d^3v \left(\frac{-\partial f_0}{\partial \epsilon}\right) \frac{\vec{v}\,\vec{v}}{\tau^{-1} - i\omega + i\vec{k}\cdot\vec{v}} \tag{3.13}$$

and

$$\vec{D} = \frac{2}{3}\frac{\epsilon_F}{N_0} \int d^3v\, \frac{\tau^{-1}}{\omega}\left(\frac{-\partial f_0}{\partial \epsilon}\right) \frac{\vec{v}}{\tau^{-1} - i\omega + i\vec{k}\cdot\vec{v}} . \tag{3.14}$$

To reduce (3.12) to a simple conductivity form

$$\vec{J}_1 = \vec{\sigma}\cdot\vec{E}_1 , \tag{3.15}$$

we define the dyadic $\mathbf{D}$ as

$$\mathbf{D}(\vec{k}, \omega) = \vec{D}(\vec{k}, \omega)\vec{k} , \tag{3.16}$$

so that the equation of continuity

$$\vec{k}\cdot\vec{J}_1 + e\omega N_1 = 0 \tag{3.17}$$

may be written as

$$\mathbf{D}\cdot\vec{J}_1 = -e\omega\vec{D} N_1 . \tag{3.18}$$

Thus, the generalized conductivity tensor $\vec{\sigma}$ becomes

$$\vec{\sigma}(\vec{k}, \omega) = [I - \mathbf{D}(\vec{k}, \omega)]^{-1} \circ \vec{\sigma}'(\vec{k}, \omega) , \tag{3.19}$$

where A^{-1} represents the inverse of tensor A.

Evaluation of $\vec{\sigma}'$ *and* $\mathbf{D}$. At $T=0$ the Fermi-Dirac distribution function (3.4) becomes a step function in energy or velocity. This permits us to reduce the velocity integration to a solid-angle integration by the identity

$$\int d^3v\, G(\vec{v}) \left(-\frac{\partial f_0}{\partial \epsilon}\right) = \frac{3N_0}{4\pi m v_F^2} \int d\Omega\, G(\vec{v}_F) . \tag{3.20}$$

Thus,
$$\vec{\sigma}' = \frac{\omega_P^2}{4\pi} \frac{1}{\tau^{-1} - i\omega} \frac{3}{4\pi} \int \frac{d\Omega\,\hat{r}\hat{r}}{1 + i\vec{a}\cdot\hat{r}} \tag{3.21a}$$

and
$$\mathbf{D} = \frac{1}{\omega\tau}\frac{\vec{a}}{4\pi} \int \frac{d\Omega\,\hat{r}}{1 + i\vec{a}\cdot\hat{r}} , \tag{3.21b}$$

where $\hat{r}$ is a unit radial vector in spherical coordinates and

$$\vec{a} = \frac{\vec{k} v_F}{\tau^{-1} - i\omega} . \tag{3.22}$$

The preceding discussion has up to now been quite general, applicable to both homogeneous and inhomogeneous waves. But to evaluate the integrals, the direction of $\vec{k}$ or $\vec{a}$ must be specified, and if a single direction is given for $\vec{k}$, as is usually the practice, the results apply only to homogeneous waves. To include inhomogeneous waves we choose $\vec{k}$ to lie in the x-z plane with the real part of $\vec{k}$ in the z direction, i.e., $\vec{k}$, or $\vec{a}$, possesses only x and z components. Evaluating the integrals under these conditions (Appendix), we find the components of $\vec{\sigma}'$ to be

$$\sigma'_{xx} = \frac{A}{a^2}\left(\frac{a_z^2(1+a^2) - 2a_x^2}{a^3}\tan^{-1}a - \frac{a_z^2 - 2a_x^2}{a^2}\right) , \tag{3.23a}$$

$$\sigma'_{yy} = \left(\frac{A}{a^2}\right)\left(\frac{(1+a^2)}{a}\tan^{-1}a - 1\right) , \tag{3.23b}$$

$$\sigma'_{zz}=\frac{A}{a^2}\left(\frac{2a_z^2-a_x^2}{a^2}-\frac{2a_z^2-a_x^2(1+a^2)}{a^3}\tan^{-1}a\right) , \quad (3.23c)$$

$$\sigma'_{xz}=\sigma'_{zx}=\left(\frac{A}{a^2}\right)\left(\frac{a_x a_z}{a^2}\right)\left(\frac{3-(3+a^2)}{a\tan^{-1}a}\right) ,$$

where $A=3\omega_P^2/8\pi(\tau^{-1}-i\omega)$, (3.24)

$$a^2=\vec{a}\cdot\vec{a}=a_x^2+a_z^2 , \quad (3.25)$$

and the diffusion tensor to be

$$\mathbf{D}=(-i/\omega\tau)(\vec{a}\vec{a}/a^2)(1-\tan^{-1}a/a) . \quad (3.26)$$

After some tedious but straightforward algebra, the components of the total conductivity $\vec{\sigma}$ may be written

$$\sigma_{xx}=\sigma_T+(k_x^2/k^2)\sigma_M , \quad (3.27a)$$

$$\sigma_{yy}=\sigma_T , \quad (3.27b)$$

$$\sigma_{zz}=\sigma_L-(k_x^2/k^2)\sigma_M , \quad (3.27c)$$

and $\sigma_{xy}=\sigma_{zx}=(k_x k_z/k^2)\sigma_M$, (3.27d)

where $\sigma_T=(A/a^2)\{[(1+a^2)/a]\tan^{-1}a-1\}$ (3.28)

is the usual transverse conductivity,

$$\sigma_L=\frac{A}{2a^2}\left(1-\frac{\tan^{-1}a}{a}\right)\left[1+\frac{i}{\omega\tau}\left(1-\frac{\tan^{-1}a}{a}\right)\right]^{-1} \quad (3.29)$$

is the longitudinal conductivity, and

$$\sigma_M=\sigma_L-\sigma_T . \quad (3.30)$$

The difference between the conductivities for the inhomogeneous wave above and that for the homogeneous wave is primarily in the factor σ_M, since in the homogeneous case $\mathrm{Im}(\vec{k})$ is in the same direction as $\mathrm{Re}(\vec{k})$ and therefore $k_x=0$. We again emphasize that for an inhomogeneous wave the direction of $\vec{k}$ has no meaning; only $\mathrm{Re}(\vec{k})$ and $\mathrm{Im}(\vec{k})$ have physically meaningful directions. Otherwise, it would appear that the conductivity tensor (3.27) could be obtained from the usual diagonal expression for homogeneous waves by simply rotating the coordinate system about the y axis.

Dispersion relations. The dispersion relations for electric waves in a general medium are found by requiring the wave fields to satisfy Maxwell's equations, which may be put in the form

$$[(c/\omega)^2\vec{k}\cdot\vec{k}I-(c/\omega)^2\vec{k}\vec{k}-\vec{\epsilon}(\vec{k},\omega)]\cdot\vec{E}_1=0, \quad (3.31)$$

where $\vec{\epsilon}=I+(4\pi i/\omega)\vec{\sigma}$ (3.32)

is the dielectric tensor of the medium. So for a nontrivial solution to (3.31) the secular equations

$$\det[(c/\omega)^2k^2I-(c/\omega)^2\vec{k}\vec{k}-\vec{\epsilon}]=0 \quad (3.33)$$

must be satisfied, giving the desired dispersion relations.

Using the $\vec{\sigma}$ we have just calculated, the components of $\vec{\epsilon}$ for inhomogeneous waves in a degenerate plasma[19] are

$$\epsilon_{xx}=\epsilon_T+(k_x^2/k^2)\epsilon_M , \quad (3.34a)$$

$$\epsilon_{yy}=\epsilon_T , \quad (3.34b)$$

$$\epsilon_{zz}=\epsilon_L-(k_x^2/k^2)\epsilon_M , \quad (3.34c)$$

$$\epsilon_{xz}=\epsilon_{zx}=(k_x k_z/k^2)\epsilon_M , \quad (3.34d)$$

where $\epsilon_T=1-\dfrac{\omega_P^2}{\omega(\omega+i\tau^{-1})}\dfrac{3}{2a^2}\left(\dfrac{1+a^2}{a}\tan^{-1}a-1\right)$,

$$\epsilon_L=1-\frac{\omega_P^2}{\omega(\omega+i\tau^{-1})}\frac{3}{a^2}\left(1-\frac{\tan^{-1}a}{a}\right) \quad (3.35)$$

$$\times\left[1+\frac{i}{\omega\tau}\left(1-\frac{\tan^{-1}a}{a}\right)\right]^{-1} , \quad (3.36)$$

$$\epsilon_M=\epsilon_L-\epsilon_T , \quad (3.37)$$

and $a^2=-\vec{k}\cdot\vec{k}v_F^2/(\omega+i\tau^{-1})^2$. (3.38)

If the waves were homogeneous, (3.31) and (3.34) would give three dispersion relations: two transverse solutions

$$(c/\omega)^2\vec{k}_T\cdot\vec{k}_T=\epsilon_T(\vec{k}_T,\omega) \quad (3.39)$$

associated with $\vec{E}$ polarized in the x and y directions, i.e., EM waves; and one longitudinal solution

$$\epsilon_L(\vec{k}_L,\omega)=0 \quad (3.40)$$

associated with $\vec{E}$ polarized in the z direction, i.e., a polarization wave.

For inhomogeneous waves the y-polarized solution is unchanged, but the x- and z-polarized solutions are now coupled. But a careful analysis of the coupled equation shows that the solution is still separable into dispersion relations of two noninteracting waves. One is an EM wave with dispersion (3.39) and polarization determined by

$$k_xE_x+k_zE_z=\vec{k}_T\cdot\vec{E}_T=0 , \quad (3.41)$$

and the other is a polarization wave with dispersion (3.40) and polarization determined by

$$k_xE_z-k_zE_x=\vec{k}_L\times\vec{E}_L=0 . \quad (3.42)$$

To summarize, the dispersion relations for inhomogeneous waves are formally identical to those for homogeneous waves, the polarization of the waves being determined by the more general conditions that $\vec{E}$ of the EM wave is divergence-free [Eq. (3.41)] and $\vec{E}$ of the polarization wave is irrotational [Eq. (3.42)].

IV. APPLICATION TO METAL PLASMA

In Sec. III, we calculated the dispersion relations for EM and polarization waves in an unbounded plasma, described by an electron gas in a uniform positive background. Even with inhomogeneous waves in a collisional plasma, we found these two waves to be independent of each other, at least in the unbounded plasma case. We now assume that this result can be applied to a bounded plasma and that the bulk dispersion relations are applicable right up to the boundary surface. Since the first assumption will be treated in detail in another paper,[20] we will only note here that, certainly, after a distance of a mean free path from the surface, the electrons lose all knowledge of the boundary. The second assumption depends partially on the first and on the length of the transition region in which the plasma density reaches its bulk value. In a metal plasma this transition length is only a few angstroms, and so presents no problems. We shall later see in connection with the plasma capacitor problem that this is not the case in classical gas plasmas, where such sharp boundaries do not exist.

Although a metal can be approximated by an electron gas in a uniform positive background, care must be used since this lattice background can interact quite strongly with the conduction electrons. The difficulty lies with the valence band electrons whose excitation energies (to the conduction band) are of the same order of magnitude as the plasmon energy $\hbar\omega_p$. In our earlier report[21] we applied the free-electron dispersion relations to silver, because experimentally it is one of the simpler metals to work with. To include the effects of the lattice, i.e., bound electrons, we assumed a frequency-independent dielectric ϵ_l,[22] which shifted the plasma frequency to its experimentally measured value. But this is a gross oversimplification, since at the relevant frequencies the d-band electrons are excited causing a resonant response in the lattice dielectric.[23]

A more reasonable choice for a free-electron-like metal is one of the alkali metals, since the valence band transition occurs below ω_p.[24] Because the k electron transition occurs at frequencies an order of magnitude higher, the effect of the lattice can be approximated by a constant dielectric or an "optical mass" (which is nearly unity).

Assuming one electron per atom, the electron densities for sodium and potassium are, respectively, 2.54×10^{22} cm^{-3} and 1.34×10^{22} cm^{-3}. Using these density values, we have computed the corresponding Fermi velocities, plasma frequencies, and plasma wavelengths and presented them in Table I together with the measured value of λ_p.[24,25] Beside the Fermi velocity, the only other parameter that enters into the calculation of the dispersion relations is the collision time τ. Since experimental values of τ are not available for thin films of sodium and potassium, we assumed several constant values ranging from $\omega_p\tau = 500$ to 50. Using these parameters, we numerically calculated the dielectric components (3.35) and (3.36), solved the dispersion relations (3.39) and (3.40), and finally computed the Fresnel equations. For simplicity we assumed that the medium bounding the metal is a vacuum so that the parameters α, β, and γ in the Fresnel equations become

TABLE I. Plasma parameters for Na and K.

Metal	Calculated			Measured
	v_F(cm/sec)	ω_p(1/sec)	λ_p(Å)	λ(Å)
Na	1.07×10^8	8.97×10^{15}	2100	2180
K	8.52×10^7	6.52×10^{15}	2890	3260

$$\alpha = \epsilon_T(\vec{k}_T, \omega)\cos\theta \quad , \tag{4.1a}$$

$$\beta = (\epsilon_T - \sin^2\theta)^{1/2} \quad , \tag{4.1b}$$

$$\gamma = [\sin^2\theta(1-\epsilon_T)][K_L^2 - \sin^2\theta]^{-1/2} \quad , \tag{4.1c}$$

where θ is the angle of incidence.

A. Reflection by an Infinitely Thick Slab

Let us compare the reflection of p-polarized light at a single plane surface, i.e., by a semi-infinite slab of metal, as predicted by the classical and the new Fresnel equations. In Figs. 4 and 5, we have plotted $\log_{10}(1-R)$, where R is the reflectance, for several angles of incidence and different collision frequencies.[26] While it appears that the effect of the longitudinal plasma wave becomes more pronounced with larger angles of incidence and smaller collision frequencies, this is somewhat misleading because of the inverted logarithmic scale (an increase on the vertical scale corresponds to an increase in the reflectance, i.e., approaching closer to unity). For example, in Fig. 5 for $\omega_p\tau = 500$ and above ω_p, the classical reflectance differs from unity by less than 10^{-5}, while our equations predict nearly 10^{-4}. Since the reflectance must be measured to an accuracy of one part in 10^5 or better, it seems doubtful whether this effect of the longitudinal plasma wave may be observed by present experimental methods. The situation, however, seems more promising in the case of thin metal films.

B. Thin Metal Films

At the end of Sec. II we found that when the wavelength of the transmitted EM wave λ_T is much larger than the wavelength of the polarization wave λ_L, multiple-reflection resonances of the polarization wave may be observed, provided the film thickness d is chosen so that

$$\lambda_T > d > \lambda_L \quad . \tag{4.2}$$

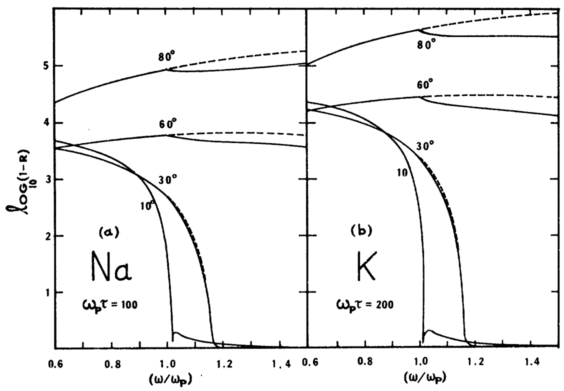

FIG. 4. Comparison of the reflectance (R) of p-polarized light predicted by the classical Fresnel equations (dashed curve) and the new equations (solid curve) which include the effect of polarization waves. The curves give $\log_{10}(1-R)$ versus frequency for several angles of incidence. They were numerically computed for a free-electron-like plasma with (a) $v_F = 1.07\times10^8$ cm/sec (corresponding to the Fermi velocity of Na) and $\omega_p\tau = 100$, and (b) $v_F = 8.52\times10^7$ cm/sec (corresponding to K) and $\omega_p\tau = 200$.

This condition may be satisfied in very thin metal films at frequencies above but near ω_p.

Neglecting damping due to collisions, the dispersion relations for a free-electron-like plasma near ω_p may be approximated by

$$K_T^2 = (\lambda/\lambda_T)^2 \simeq 1-(\omega_p/\omega)^2 \quad , \tag{4.3a}$$

$$K_L^2 = (\lambda/\lambda_L)^2 \simeq \tfrac{5}{3}\,(c/v_F)^2[(\omega/\omega_p)^2-1] \ , \tag{4.3b}$$

so that $$(\lambda_L/\lambda_T)^2 \simeq \tfrac{3}{5}(v_F/c)^2(\omega_p/\omega)^2 \quad . \tag{4.4}$$

Thus, λ_T is at least a couple orders of magnitude larger than λ_L. Since, for $\omega > \omega_p$, $\lambda_T > \lambda$, the upper limit on the film thickness is

$$d < \lambda_p \quad . \tag{4.5}$$

From Table I, we find that d has to be a few hundred angstroms or less for sodium or potassium.

Figs. 6–11 present numerical results for unbacked (vacuum on both sides) potassium foils of various thicknesses, for different angles of incidence, and several collision frequencies. Figures 6–8 compare the transmittance, reflectance and absorptance predicted by the classical equations (2.45) and our Eqs. (2.42) for a 52-Å-thick potassium film. The large main resonance centered at ω_p, and predicted by both the classical and our theories, appears to be due to the refraction of the EM wave in the foil, so that it propagates nearly parallel to the surface. Such a wave will have the appearance of a surface wave, the propagation being almost along the surface and the electric field nearly normal to it. In addition to this main resonance, our results (solid line) clearly show smaller structure due to the multiple-reflection resonance. These resonances are especially apparent in absorptance spectra for large angles of incidence. Note the first resonance occurs at $\omega > \omega_p$, corresponding to $\lambda_L = 2d$ and, therefore, shifts to larger ω for thinner films (Fig. 10).

Although the main resonance at ω_p has been observed in several metals[3,4,27] no one has observed the finer structure. One reason for this may be that real metal effects, e.g., interband transitions, may mask this structure. But while this may be the case with metals such as silver, the alkali metals which we have considered in this paper should not have this limitation.[28] A more probable reason may be that the metal films used in these experiments did not have the required smooth parallel surfaces. Since the standing waves require coherent reflection, the surfaces must be smooth and parallel to within a fraction of λ_L, i.e., to within about 10 Å. But recent work has shown that these surfaces are quite rough[29] and the roughness seems to be responsible for the reradiation discovered by Brambring and Raether.[30] A possible remedy for this surface roughness and nonparallelism may be sandwiching the metal between two optically flat quartz plates. Such a procedure would be especially advantageous with the alkali metals which are too soft for making unsupported films.

In addition to verifying the macroscopic theory of polarization waves that we have presented here, the thin film experiment described above would shed some useful information on plasma waves such as their dispersion relation. In particular, it may provide the first observation of quantum effects on the plasma wave when $k_F\lambda_L < 1$ [20] and give a measure of the critical cutoff length λ_D in solid state plasmas.

V. CONCLUDING REMARKS

In this paper, we have developed a macroscopic

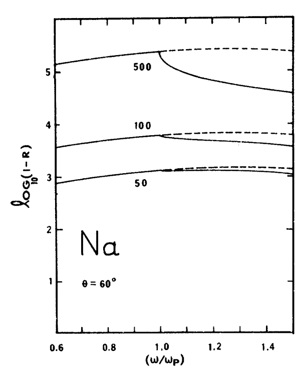

FIG. 5. $\log_{10}(1-R)$ spectrum of Na at 60° incidence for several collision frequencies.

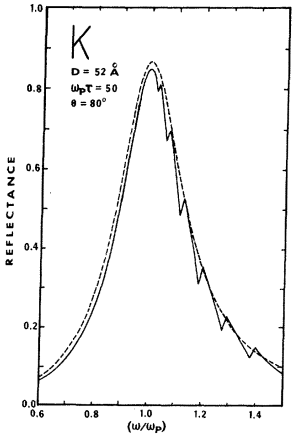

FIG. 6. Reflectance of p-polarized light by a 52-Å-potassium foil for an 80° angle of incidence and a collision frequency corresponding to $\omega_p\tau = 50$. The curves were obtained numerically using the classical Fresnel equations (dashed curve) and the new equations (solid curve).

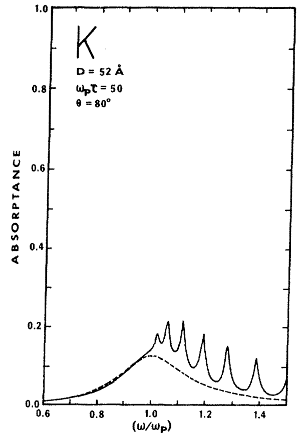

FIG. 8. Absorptance of p-polarized light by a 52-Å-potassium foil.

(Fresnel equations) theory of optical excitation of polarization waves in isotropic homogeneous media, and applied it to the problem of exciting bulk plasma density waves (plasmons) in metals. The effect of these waves in semi-infinite metal slabs was found to be very small, making them difficult to observe. But in extremely thin metal films their presence may be observed as resonances when standing waves of these plasmons are set up.

Because the interaction of EM waves or oscillatory E fields with bounded plasmas is as old as plasma physics itself, a brief sketch of the relation between the present problem and similar problems encountered in classical gas plasmas is given below.

A. Plasma Capacitor and Tonks-Dattner Resonances

While to our knowledge we were the first to suggest that bulk plasma waves may be resonantly excited in metal films,[5] the phenomena of standing longitudinal waves in a plasma slab was considered earlier in connection with the plasma capacitor problem.[31] As the name suggests, the problem involved a parallel-plate capacitor filled with a homogeneous isotropic warm plasma. Since the EM wave plays no role, the longitudinal plasma waves can be studied directly and the problem easily lends itself to a normal mode analysis.[32] At frequencies corresponding to standing waves[33]

$$\omega_n^2 = \omega_p^2 + (3kT/m)(n\pi/d)^2 \quad , \tag{5.1}$$

the capacitor responds resonantly, the absorptance being similar to those in Fig. 10, but without the broad absorption centered at ω_p.

The plasma capacitor problem arose out of efforts to explain the Tonks-Dattner resonances[34]: Resonant scattering of EM radiation by discharge tube plasmas at discrete series of frequencies related to ω_p. Even though the standing-wave mechanism was correct, the capacitor theory, or more precisely its cylindrical analog, failed to explain

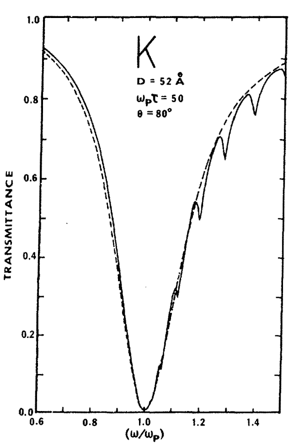

FIG. 7. Transmittance of p-polarized light by a 52-Å-potassium foil.

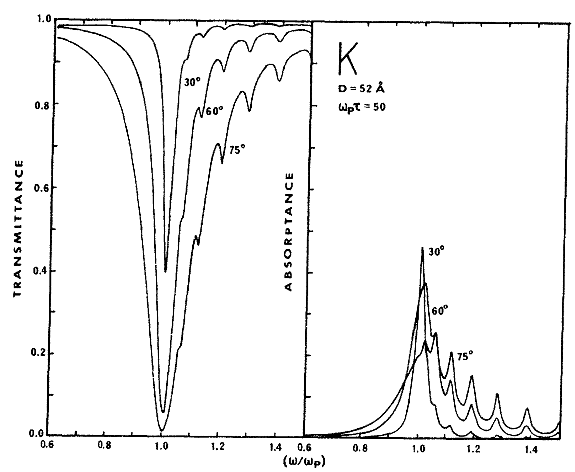

FIG. 9. Variation of transmittance and absorptance of p-polarized light by a 52-Å-potassium foil with the angle of incidence.

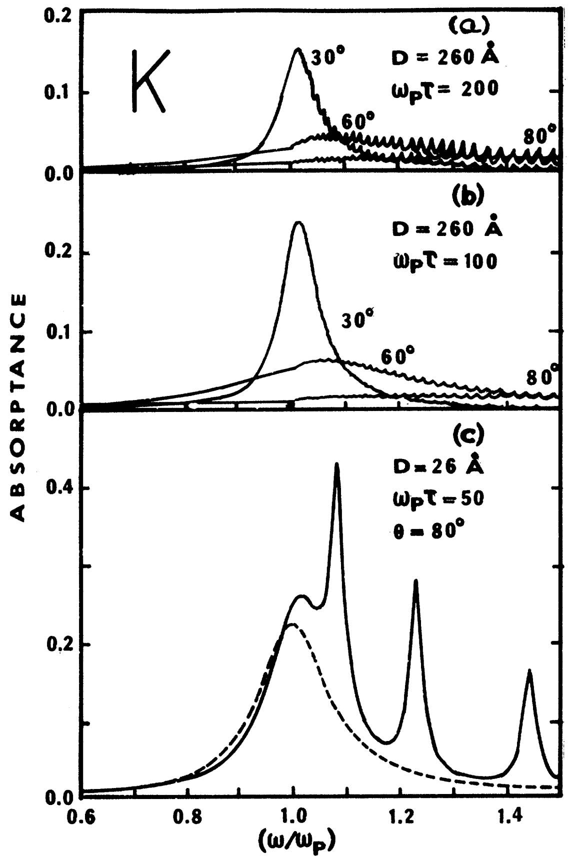

FIG. 10. Absorptance of p-polarized light by potassium foils; 260-Å foil for several angles of incidence and $\omega_p\tau$ = 200 (a) and 100 (b). Comparison of the classical and the new absorption by a 26-Å-potassium foil (c).

these resonances because such plasmas cannot satisfy the assumption of homogeneity. Instead the density profile, and therefore ω_p, monotonically decreases from its maximum value at the center and, thus, a plasma wave of frequency ω will originally propagate in a region where $\omega > \omega_p(r)$ until it reaches a critical distance where $\omega = \omega_p \times (r_c) < (\omega_p)_{max}$ and is then reflected. Standing waves of varying wavelength are thus created between the "edge" of the plasma and the plasma and the critical point r_c, and the resulting resonance spectrum is qualitatively different from the homogeneous case.[35]

While the Tonks-Dattner resonances have been studied primarily in laboratory plasmas, there has also been considerable interest in plasma-wave resonances in natural plasmas. Specifically, recent studies by topside sounders[36] have revealed resonances in the ionosphere which are related to ω_p but with the additional complication of a cyclotron frequency due to the presence of earth's magnetic field. Although these resonances are as yet not completely understood, they appear to involve longitudinal or electrostatic plasma waves.[37]

B. Other Polarization Waves

Although this paper has primarily considered longitudinal plasma waves in metals, the theory developed in Sec. II is applicable to any polarization waves in a homogeneous isotropic medium. Thus, for example, the theory can be applied to optical excitation of longitudinal optical (LO) phonons in a polar crystal, such as an alkali halide.

In an experiment analogous to the plasma resonance experiment,[4] Berreman[8] measured the LO frequency in thin foils of LiF. But as with plasma waves, the LO phonon has a dispersion relation, shown in Fig. 12, so that at frequencies below ω_{LO} polarization waves of finite wavelength will propa-

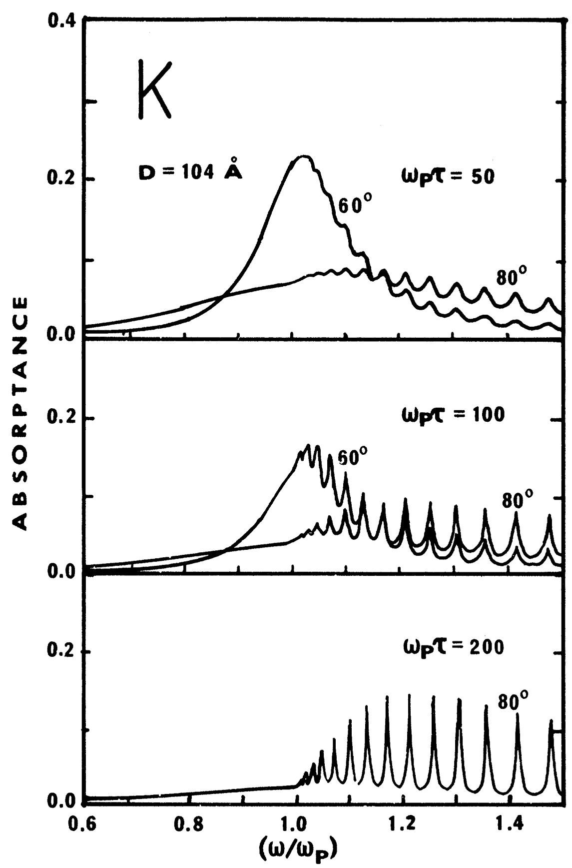

FIG. 11. Absorptance of p-polarized light by a 104-Å-potassium foil for several angles of incidence and several collision frequencies.

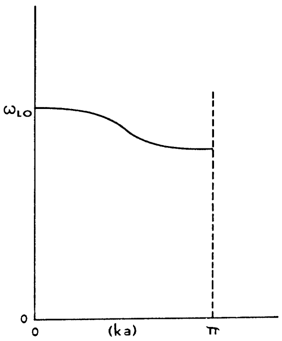

FIG. 12. Dispersion relation of a LO phonon.

gate. Thus, we would expect smaller absorption structure below ω_{LO} corresponding to standing polarization waves. Such a measurement may be more difficult to perform than the metal plasma resonance since it requires thin single-crystal films.[38]

ACKNOWLEDGMENTS

This research was partially supported by National Science Foundation Grant No. GP 5098 and completed while one of us (A. R. M.) held a National Academy of Sciences/National Research Council Postdoctoral Associateship with the ESSA Research Laboratories. A. Melnyk wishes to thank Dr. J. Weinstock of the Plasma Physics Laboratory for his hospitality, and all the members of the laboratory for helpful conversations. Finally, he wishes to thank Dr. W. M. Leavens for bringing to his attention the Tonks-Dattner resonances.

APPENDIX: EVALUATION OF $\mathbf{T}$ AND $\vec{V}$

In the spherical coordinate system (Fig. 13), the integrals

$$\mathbf{T} = \int \frac{d\Omega\, \hat{r}\hat{r}}{1 + i\vec{a}\cdot\hat{\vec{r}}} \quad , \tag{A1}$$

$$\vec{V} = \int \frac{d\Omega \hat{r}}{1 + i\vec{a}\cdot\hat{r}} \quad , \tag{A2}$$

become

$$\mathbf{T} = \int_0^{2\pi} d\varphi \int_{-1}^{+1} \frac{du\, \hat{r}\hat{r}}{1 + ia_z u + ia_x(1-u^2)^{1/2}\cos\varphi} \quad , \tag{A3}$$

$$\vec{V} = \int_0^{2\pi} d\varphi \int_{-1}^{+1} \frac{du\, \hat{r}}{1 + ia_z u + ia_x(1-u^2)^{1/2}\cos\varphi} \quad . \tag{A4}$$

For homogeneous waves, $a_x = 0$ and (A3) and (A4) take on a simple form which is easily evaluated

$$T_{xx} = T_{yy} = (2\pi/a_z^2)\,\{[1 + a_z^2/a_z]\tan^{-1} a_z - 1\}\,,$$

$$T_{zz} = (4\pi/a_z^2)\,(1 - \tan^{-1} a_z/a_z)\,, \tag{A5}$$

and $V_z = (4\pi/ia_z)\,(1 - \tan^{-1} a_z/a_z)\,,$ (A6)

all other components being zero.

To evaluate (A3) and (A4) for the general inhomogeneous wave case, we need the value of the integral

$$I = \int_0^{\pi} \frac{dx}{v + w\cos x} \quad . \tag{A7}$$

The indefinite integral of (A7) has the following values[39]:

$$(v^2 - w^2)^{-1/2} \arccos\left(\frac{w + v\cos x}{v + w\cos x}\right), \quad v^2 > w^2$$

$$v^{-1}\tan(\tfrac{1}{2}x), \qquad v = w$$

$$-v^{-1}\cot(\tfrac{1}{2}x), \qquad v = -w$$

and $$(w^2 - v^2)^{-1/2}\ln\left(\frac{w + v\cos x + (w^2 - v^2)^{1/2}\sin x}{v + w\cos x}\right), \qquad w^2 > v^2 \quad .$$

Therefore, the integral (A7) is defined as long as v^2 is unequal to w^2

$$I = \pi/(v^2 - w^2)^{1/2}\,, \qquad v^2 \neq w^2 \quad . \tag{A8}$$

From (A8), we find

$$\int_0^{2\pi} \frac{d\varphi}{v + w\cos\varphi} = \frac{2\pi}{(v^2 - w^2)^{1/2}}\,,$$

$$\int_0^{2\pi} \frac{\cos\varphi\, d\varphi}{v + w\cos\varphi} = \frac{2\pi}{w}\; 1 - \frac{v}{(v^2 - w^2)^{1/2}}\,,$$

$$\int_0^{2\pi} \frac{\cos^2\varphi\, d\varphi}{v + w\cos\varphi} = \frac{2\pi}{w^2}\left(\frac{v^2}{(v^2 - w^2)^{1/2}} - v\right), \tag{A9}$$

$$\int_0^{2\pi} \frac{\sin\varphi\cos^n\varphi\, d\varphi}{v + w\cos\varphi} = 0\,,$$

$$\int_0^{2\pi} \frac{\sin^2\varphi\, d\varphi}{v + w\cos\varphi} = \frac{2\pi}{w^2}\;(v - (v^2 - w^2)^{1/2}) \quad .$$

We now extend the integrals (A9) to include complex v and w and state without proof that (A9) exists as long as the absolute value of U does not vanish,

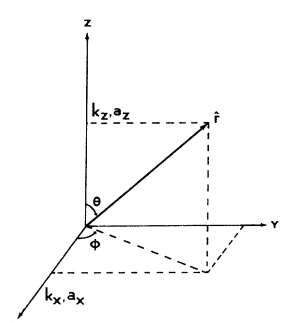

FIG. 13. Coordinate system defining direction of tensor components.

where

$$U = v^2 - w^2 = 1 + a_x^2 + 2ia_z u - (a_x^2 + a_z^2)u^2 \ . \tag{A10}$$

The value of the first three integrals of the form

$$I_n = \int_{-1}^{+1} \frac{u^n du}{(U)^{1/2}}$$

can be expressed as

$$I_0 = 2\tan^{-1}a/a \ ,$$

$$I_1 = -2i(a_z/a^2)\,[1 - \tan^{-1}a/a]\,, \tag{A11}$$

$$I_3 = [(2a_z^2 - a_x^2)/a^4] - ([2a_z^2 - a_x^2(1+a^2)]/a^5)\tan^{-1}a \ ,$$

$$\text{where } a^2 = \vec{a}\cdot\vec{a} = a_x^2 + a_z^2 \ . \tag{A12}$$

With (A9) and (A11), the desired integrals may now be evaluated

$$T_{xx} = 2\pi\left[\left(\frac{a_z^2(1+a^2) - 2a_x^2}{a^5}\right)\tan^{-1}a - \left(\frac{a_z^2 - 2a_x^2}{a^4}\right)\right] \ ,$$

$$T_{xz} = T_{zx} = 2\pi(a_x a_z/a^4)\,[3 - (3 + a^2/a)\tan^{-1}a\,] \ ,$$

$$T_{yy} = (2\pi/a^2)\,[(1 + a^2/a)\tan^{-1}a - 1\,] \tag{A13}$$

$$T_{zz} = 2\pi\left[\left(\frac{2a_z^2 - a_x^2}{a^4}\right) - \left(\frac{2a_z^2 - a_x^2(1+a^2)}{a^5}\right)\tan^{-1}a\right],$$

$$T_{xy} = T_{yx} = T_{yz} = T_{zy} = 0 \ ,$$

and

$$V = \frac{4\pi}{i}\,\frac{a_x}{a^2}\left(1 - \frac{\tan^{-1}a}{a}\right), \quad V_y = 0, \quad V_z = \frac{4\pi}{i}\,\frac{a_z}{a^2}\left(1 - \frac{\tan^{-1}a}{a}\right), \tag{A14}$$

with the condition that

$$|a| \neq 1 \ . \tag{A15}$$

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*Portion of work based on a thesis submitted by A. R. M. in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Michigan State University.

[1]R. A. Ferrell, Phys. Rev. 111, 1214 (1958).

[2]R. A. Ferrell and E. A. Stern, Am. J. Phys. 30, 810 (1962).

[3]S. Yamaguchi, J. Phys. Soc. Japan 17, 1172 (1962).

[4]A. J. McAlister and E. A. Stern, Phys. Rev. 132, 1599 (1963).

[5]Aside from such phenomena as the anomalous skin effect, see G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) A195, 336 (1948).

[6]By polarization waves we mean longitudinal electrostatic waves associated with oscillating charge densities $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$.

[7]D. W. Berreman, Phys. Rev. 130, 2193 (1963).

[8]It might be supposed that, since the use of local dielectric functions $\epsilon(\omega)$ by Refs. 4 and 7 in the classical Fresnel equations led to surface polarization waves, the use of a nonlocal dielectric function $\epsilon(k,\omega)$ would lead to bulk polarization waves. But this argument only underlines the inadequacy of the classical equations since the transverse and longitudinal nonlocal dielectric functions are no longer equal and only one dielectric function (the transverse one) appears in the classical equations.

[9]For example, J. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), p. 502.

[10]Since we are interested only in nonmagnetic materials for which $B = H$, only H will appear from now on.

[11]F. Sauter, Z. Physik 203, 488 (1967); A. R. Melnyk, Ph. D. thesis, Michigan State University, 1967 (unpublished).

[12]We consider here only the linearly related cases, i.e., where the current is proportional to the electric field.

[13]One distinction that can be made between the two currents is that the real polarization current is out of phase with the electric field, while the real conduction current is in phase, and, therefore, only the latter dissipates energy from the field.

[14]Because Sauter (Ref. 11) makes, what we consider, an arbitrary distinction between polarization and conduction currents, his boundary conditions differ from ours.

[15]Note, the expressions for T_p differ slightly from the usual classical Fresnel equations. This is because the usual definition of T is E_T/E_0, whereas, we have used $(K_0 \times E_T)/(K_T \times E_0)$; thus, our expressions may be converted to the usual by multiplying T_p by $K_T/K_0 = (\epsilon_T/\epsilon_0)^{1/2}$

[16]Similar but more restricted equations have been obtained by A. M. Fedorchenko {Zh. Tekhn. Fiz. 32, 589 (1962); 36, 1327 (1966) [Soviet Phys. Tech. Phys. 7, 428 (1962); 11, 992 (1967)]} in calculating the conversation of transverse electromagnetic waves into longitudinal waves at a dielectric-plasma plane interface. In Fedorchenko's first paper, he pointed out that the usual boundary conditions specifying continuity of tangential E and H are insufficient to solve the problems, and he added the condition that the normal component of all plasma charges velocity vanishes at the boundary. His results were in general incorrect, however, because this last boundary condition applies only to a plasma-vacuum interface, since the existence of a dielectric presupposes polarization charges which can provide a nonvanishing value for the normal component of current density on the plasma-dielectric interface. But at a plasma-vacuum boundary, i.e., $\epsilon_0 = 1$, the normal component of the current density of charge motion vanishes, and our results applied to homogeneous waves coincide with Fedorchenko's 1962 results. Although he corrected his choice of boundary conditions in the 1967 paper to effectively include the effect of polarization currents in the dielectric, his published results differ from ours. We ascribe this to a misprint, since he claims in the second paper that the results of both papers coincide for a plasma-vacuum boundary, but in fact they do not.

[17]Another interesting case is the reflection by a medium consisting of a collection of localized oscillators. At the resonance frequency, the oscillators will appear as a polarization wave with an infinite wavelength, and so γ will be very large. But because at resonance, ϵ_T and α vanish, both the new and the classical equations give total reflection.

[18]O. S. Heavens, *Optical Properties of Thin Solid Films* (Academic, New York, 1955), p. 56.

[19]These equations are applicable for wavelengths greater than the Debye screening length λ_D or the electron wavelength $\lambda_F = h/mv_F$. If $\lambda \lesssim \lambda_F$, quantum-mechanical expressions for the dielectric must be used, e.g., J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 28, No. 8 (1954).

[20]A. R. Melnyk and M. J. Harrison, following paper, Phys. Rev. B1, 000 (1970).

[21]A. R. Melnyk and M. J. Harrison, Phys. Rev. Letters 31, 85 (1968).

[22]The optical behavior of any metal may be described by a free-electron gas imbedded in a frequency-dependent dielectric due to bound electrons. Except at resonant frequencies, e.g., frequencies corresponding to interband transitions, this lattice dielectric is a slowly varying function of frequency.

[23]H. Ehrenreich and H. R. Phillip, Phys. Rev. 128, 1622 (1962).

[24]J. C. Sutherland, E. T. Arakawa, and R. N. Hamm, J. Opt. Soc. Am. 57, 645 (1967); 58, 1080 (1968).

[25]J. Bösenberg, Phys. Letters 26A, 74 (1967).

[26]Similar results were found by F. Fortsmann, Z. Physik 203, 495 (1967), using a microscopic theory. His results become identical to ours if our expression for R is linearized in $\gamma/(\alpha+\beta)$, i.e.,

$$R = \left| \frac{\alpha-\beta-\gamma}{\alpha+\beta+\gamma} \right|^2 \simeq \left| \left(\frac{\alpha-\beta}{\alpha+\beta}\right) 1 - \left(\frac{2\gamma}{\alpha+\beta}\right) \right|^2 .$$

[27]R. Fleishmann, Z. Physik 174, 104 (1963); A. Ejiri and T. Sasaki, J. Phys. Soc. Japan 20, 876 (1965); J. Brambring, Z. Physik 200, 186 (1967).

[28]Real metal effects on the multiple-reflection resonance are now being studied.

[29]P. Schreiber, Z. Physik 211, 257 (1968).

[30]J. Brambring and H. Raether, Phys. Rev. Letters 15, 882 (1965).

[31]R. B. Hall, Am. J. Phys. 31, 696 (1963); F. C. Shure, J. Nucl. Energy C6, 1 (1964).

[32]The normal mode method has been recently applied to the plasma-slab problem with an EM wave: F. L. Hinton, Phys. Fluids 10, 2408 (1967); J. J. Bowman and V. H. Weston, *ibid.* 11, 601 (1967); W. E. Jones, K. L. Kliewer, and R. Fuchs (unpublished). Jones *et al.* considered a metal foil and found, besides the multiple-reflection resonance, additional anomalous absorption below ω_p. We thank Dr. Jones, Dr. Kliewer, and Dr. Fuchs for sending us a copy of their paper before publication.

[33]The factor (kT/m) of a classical thermal plasma corresponds to $\frac{1}{5}v_F^2$ of a solid-state quantum plasma.

[34]L. Tonks, Phys. Rev. 37, 1458 (1931); A. Dattner, Ericsson Technics 2, 309 (1957); 8, 1 (1963).

[35]J. V. Parker, J. C. Nickel, and R. W. Gould, Phys. Fluids 7, 1489 (1964).

[36]A good review of the topside sounding technique may be found in W. Calvert, Science 154, 228 (1966).

[37]A brief description of these resonances and the relevant references is given by W. Calvert (Ref. 36).

[38]Since the dispersion relation for the LO phonons is very flat near ω_{LO}, the resonances will be very closely spaced unless the films are very thin. Preliminary results seem to indicate that the required thickness may be only a few hundred angstroms.

[39]G. Petit Bois, *Table of Indefinite Integrals* (Dover, New York, 1961), pp. 121–122.